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Completeness and cut-elimination theorems for trilattice logics

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ABSTRACT

A sequent calculus L_{16} for Odintsov's Hilbert-style axiomatization L_B of a logic related to the trilattice *SIXTEEN*$_3$ of generalized truth values is introduced. The completeness theorem w.r.t. a simple semantics for L_{16} is proved using Maehara's decomposition method that simultaneously derives the cut-elimination theorem for L_{16}. A first-order extension F_{16} of L_{16} and its semantics are also introduced. The completeness and cut-elimination theorems for F_{16} are proved using Schütte's method.

1. Introduction

The present paper deals with certain logics related to the trilattice *SIXTEEN*$_3$ of generalized truth values. At first sight, this investigation might appear to be devoted to problems that are rather peripheral to mainstream formal logic. At closer inspection, however, this it not the case. The first impression might be due to the fact that the trilattice *SIXTEEN*$_3$ has been introduced into the literature only recently, namely in [19], and that the logics under consideration in the present paper have been defined even more recently, namely in [15]. We cannot unfold here the motivation for studying *SIXTEEN*$_3$ in any detail, because this would have to include a historical and philosophical discussion of the notion of a generalized truth value, generalized truth values being subsets of an already given set of truth values. Such a general discussion and motivation can be found in [21]. We may, however, point out that the logical study of trilattices builds upon a mature mathematical theory, namely the theory of bilattices, see, for example, [1–4,8–12,17]. In particular, the trilattice *SIXTEEN*$_3$ is a natural and straightforward generalization of the smallest non-trivial bilattice *FOUR*$_2$. This bilattice is defined on the powerset of the set of classical truth values T and F. The four-valued logic of *FOUR*$_2$ with $\{T\}$ and $\{T, F\}$ as designated values is known as Dunn's and Belnap's (useful) four-valued logic or as first-degree entailment logic, FDE. FDE has found many applications in, for example, many-valued symbolic model-checking, the semantics of logic programs, intelligent tutoring systems, inconsistency-tolerant description logics, and generalizations of algebras of commuting processes.

We will consider two logics related to *SIXTEEN*$_3$, namely the axiom systems L_B and L_T presented by Odintsov [15]. Since the two axiom systems L_B and L_T can be dealt with in a completely similar way, we shall focus our attention just on the system L_B. The logics L_B and L_T are of interest, among other things because they combine a set of positive connectives (related to truth) and a set of negative connectives (related to falsity), a combination which emerges naturally in the context of *SIXTEEN*$_3$, see [19,20,22,18].[1] We define a sequent calculus L_{16} for L_B, which differs from the sequent calculus GL_B for L_B defined in [14].

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[1] In [22] the present paper is referred to as an unpublished manuscript with the title "Alternative semantics for trilattice logics".

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The sequent calculus L_{16} is such that it can conveniently be shown to be strongly complete with respect to a variant of the co-ordinate valuations semantics introduced by Odintsov [15].

Classical propositional logic has both an algebraic semantics in terms of Boolean algebras and a non-algebraic semantics in terms of truth-value assignments. The trilattice logics considered in [19,20] are non-classical propositional logics with an algebraic semantics. One route to obtain a semantics of *quantified* trilattice logics would consist of defining a suitable notion of cylindric algebras for them. It may be seen as an advantage of the co-ordinate valuations semantics that it admits of a simple and straightforward extension to first-order logic. The present paper is the first paper on first-order trilattice logics. We present a cut-free, sound and complete sequent calculus F_{16} for a first-order extension of the logic L_B.

As already said, the trilattice $SIXTEEN_3$ introduced in [19] is a natural generalization of the famous bilattice $FOUR_2$. Whereas in $FOUR_2$, in addition to the information order, there is only one logical order used to define semantical consequence, in $SIXTEEN_3$ there are two logical orders, a *truth* order and a *falsity* order. Truth and falsity are thereby treated on a par as independent notions in their own right. Each of the two logical orders induces a set of logical operations and an entailment relation. The relation $\models_t$ is defined with respect to the truth order and the relation $\models_f$ is defined with respect to the falsity order. In [19] an axiomatization is presented for $\models_t$ in the language based on the truth order and for $\models_f$ in the language based on the falsity order. Moreover, the relations $\models_t$ and $\models_f$ are axiomatized in the positive language extended by falsity negation and in the negative language extended by truth negation, respectively. It was left as an open problem, however, to axiomatize the relations $\models_t$ and $\models_f$ in the full vocabulary containing both the truth and the falsity connectives. Odintsov [15] presented a sound and complete axiomatization of $\models_t$ based on the full language extended by an implication operation. The implication is interpreted as the residuum of truth conjunction with respect to the truth order.

For a detailed motivation of logics emerging from trilattices of generalized truth values we refer to [19,20,22]. The relation of these logics to many-valued logics defined from sets of designated truth values is discussed in [26]; proof-theoretic aspects are investigated in [13–15,24,25].

The present paper is structured as follows. In Section 2, we present Odintsov's axiom systems L_B and L_T and explain how they are related to the trilattice of generalized truth values $SIXTEEN_3$. In Section 3, the sequent calculus L_{16} for L_B is defined. Moreover, a variant of Odintsov's co-ordinate valuations semantics is introduced, and Maehara's method is applied to show that L_{16} is strongly sound and complete with respect to this semantics. A semantic proof of cut-elimination follows immediately. Section 4 is devoted to extending L_{16} to a first-order sequent calculus F_{16}. Using Schütte's method involving the notion of saturated sequents, F_{16} is shown to be sound and complete with respect to first-order models using four interpretation functions (according to the variation of Odintsov's co-ordinate valuations) and four corresponding satisfaction relations. Again, cut-elimination can be proved semantically.

2. Motivations and backgrounds

2.1. The trilattice $SIXTEEN_3$

$SIXTEEN_3$ is the smallest so-called Belnap trilattice [19,20,22,18]. It is based on the 16-element powerset of the powerset of the set of classical truth values $\mathbf{2} = \{T, F\}$, and it is motivated by generalizing Belnap's [5,6] idea of viewing a truth value as information that is told to a computer concerning a given proposition. Whereas the elements of $\mathbf{4} = \mathcal{P}(\mathbf{2})$ can be understood as follows:

$\mathbf{N} = \{\}$ — "told neither true nor false";
$\mathbf{F} = \{F\}$ — "told only false";
$\mathbf{T} = \{T\}$ — "told only true";
$\mathbf{B} = \{T, F\}$ — "told both true and false",

so informed computers themselves may or may not pass Belnap's generalized truth values to other computers, which thereby receive combinations of Belnap's values as information concerning a given proposition. They may, e.g., receive the information that a proposition is "both told neither true nor false and told both true and false" ($\{\emptyset, \{T, F\}\}$). It turns out that on the resulting set of values $\mathcal{P}(\mathbf{4}) = \mathbf{16}$:

1.	$\mathbf{N} = \emptyset$	9.	$\mathbf{FT} = \{\{F\}, \{T\}\}$
2.	$\boldsymbol{N} = \{\emptyset\}$	10.	$\mathbf{FB} = \{\{F\}, \{F, T\}\}$
3.	$\boldsymbol{F} = \{\{F\}\}$	11.	$\mathbf{TB} = \{\{T\}\}, \{F, T\}\}$
4.	$\boldsymbol{T} = \{\{T\}\}$	12.	$\mathbf{NFT} = \{\emptyset, \{F\}, \{T\}\}$
5.	$\boldsymbol{B} = \{\{F, T\}\}$	13.	$\mathbf{NFB} = \{\emptyset, \{F\}, \{F, T\}\}$
6.	$\mathbf{NF} = \{\emptyset, \{F\}\}$	14.	$\mathbf{NTB} = \{\emptyset, \{T\}, \{F, T\}\}$
7.	$\mathbf{NT} = \{\emptyset, \{T\}\}$	15.	$\mathbf{FTB} = \{\{F\}, \{T\}, \{F, T\}\}$
8.	$\mathbf{NB} = \{\emptyset, \{F, T\}\}$	16.	$\mathbf{A} = \{\emptyset, \{T\}, \{F\}, \{F, T\}\}$

in addition to set-inclusion as a natural information order $\leq_i$, a truth order $\leq_t$ and a falsity order $\leq_f$ can be defined (which are not inverses of each other).

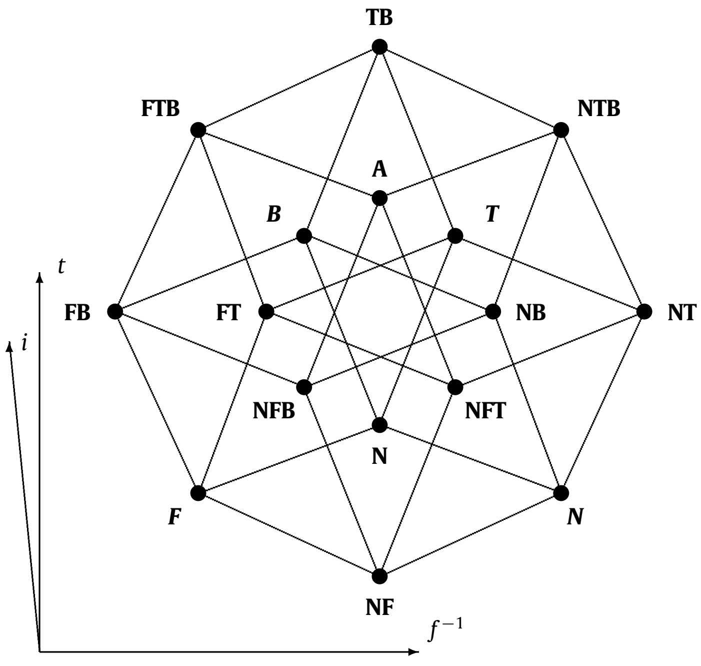

Fig. 1. Trilattice $SIXTEEN_3$.

Definition 1. For every x, y in **16**:

(1) $x \leq_i y$ iff $x \subseteq y$;
(2) $x \leq_t y$ iff $x^t \subseteq y^t$ and $y^{-t} \subseteq x^{-t}$,
where $x^t := \{y \in x \mid T \in y\}$ and $x^{-t} := \{y \in x \mid T \notin y\}$;
(3) $x \leq_f y$ iff $x^f \subseteq y^f$ and $y^{-f} \subseteq x^{-f}$,
where $x^f := \{y \in x \mid F \in y\}$ and $x^{-f} := \{y \in x \mid F \notin y\}$.

Note that the definition of the truth (falsity) order refers only to the classical value T (F) and not to the value F (T).

The three (complete) lattices $(\mathbf{16}, \leq_i)$, $(\mathbf{16}, \leq_t)$, and $(\mathbf{16}, \leq_f)$ can be combined into the *trilattice* $SIXTEEN_3 = (\mathbf{16}, \leq_i, \leq_t, \leq_f)$, see [19,22]. $SIXTEEN_3$ is depicted as a Hasse diagram in Fig. 1; alternatively, it may be presented as the algebraic structure $\langle \mathbf{16}, \sqcap_i, \sqcup_i, \sqcap_t, \sqcup_t, \sqcap_f, \sqcup_f \rangle$, where $\sqcap_\sharp$ $(\sqcup_\sharp)$ is the meet (join) with respect to $\leq_\sharp$, $\sharp \in \{i, t, f\}$. Since the "logical" relations $\leq_t$ and $\leq_f$ are treated on a par, the operations $\sqcap_t$ and $\sqcup_t$ are not privileged as interpretations of conjunction and disjunction. The operation $\sqcup_f$ may as well be regarded as a conjunction and $\sqcap_f$ as a disjunction. Therefore, the logical vocabulary can be considered to comprise a positive truth vocabulary together with a negative falsity vocabulary. Also certain unary truth and falsity operations with negation-like properties are available in $SIXTEEN_3$.

A unary operation $-_t$ $(-_f)$ on $SIXTEEN_3$ is said to be a t-inversion (f-inversion) iff the following conditions are satisfied:

1. *t-inversion* $(-_t)$:	2. *f-inversion* $(-_f)$:
(a) $x \leq_t y \Rightarrow -_t y \leq_t -_t x$;	(a) $x \leq_t y \Rightarrow -_f x \leq_t -_f y$;
(b) $x \leq_f y \Rightarrow -_t x \leq_f -_t y$;	(b) $x \leq_f y \Rightarrow -_f y \leq_f -_f x$;
(c) $x \leq_i y \Rightarrow -_t x \leq_i -_t y$;	(c) $x \leq_i y \Rightarrow -_f x \leq_i -_f y$;
(d) $-_t -_t x = x$.	(d) $-_f -_f x = x$.

A t-inversion (f-inversion) thus inverts the truth (falsity) order, leaves the other orders untouched, and is period-two. In $SIXTEEN_3$ such operations are definable as shown in Table 1. If the condition that an inversion preserves the other orderings is given up, the definition of t-inversion (f-inversion) refers only to the truth-order (falsity-order). What this suggests is that not only conjunction and disjunction, but also negation comes in two versions, because $-_t$ and $-_f$ are both natural interpretations for a negation connective. Moreover, since $x \sqcap_t y \neq x \sqcup_f y$, $x \sqcup_t y \neq x \sqcap_f y$ and $-_t x \neq -_f x$, the two logical orderings $\leq_t$ and $\leq_f$ indeed give rise to pairs of *distinct* logical operations with the same arity.

2.2. Odintsov's axiomatizations

The following list of symbols is adopted for the language used in this paper: countably many propositional variables $p_0, p_1, \ldots$, logical connectives $\rightarrow$, $\neg$, $\wedge_t$, $\vee_t$, $\wedge_f$, $\vee_f$, $\sim_t$ and $\sim_f$. The connectives $\rightarrow$, $\neg$, $\wedge_t$ and $\vee_t$ are just the classical

Table 1
t- and f-inversions in *SIXTEEN*$_3$.

x	$-_t x$	$-_f x$	x	$-_t x$	$-_f x$
N	**N**	**N**	**NB**	**FT**	**FT**
N	***T***	***F***	**FB**	**FB**	**NT**
F	***B***	***N***	**TB**	**NF**	**TB**
T	***N***	***B***	**NFT**	**NTB**	**NFB**
B	***F***	***T***	**NFB**	**FTB**	**NFT**
NF	**TB**	**NF**	**NTB**	**NFT**	**FTB**
NT	**NT**	**FB**	**FTB**	**NFB**	**NTB**
FT	**NB**	**NB**	**A**	**A**	**A**

implication, negation, conjunction and disjunction, respectively. Note that $\rightarrow$ is denoted as $\rightarrow_t$ in [15] (because it is interpreted as the residuum of the truth order in the trilattice *SIXTEEN*$_3$). The symbol $\sim_b$ is used to denote $\sim_f\sim_t$ or $\sim_t\sim_f$, and the symbol $\equiv$ is used to denote the equality of sets of symbols. Greek lower-case letters $\alpha, \beta, \ldots$ are used to denote formulas. An expression $\alpha \leftrightarrow \beta$ is an abbreviation of $(\alpha\rightarrow\beta)\wedge_t(\beta\rightarrow\alpha)$.

Odintsov's Hilbert-style axiomatizations L_{base}, L_B and L_T [15] of logics related to the trilattice *SIXTEEN*$_3$ of generalized truth values in the propositional language based on $\{\sim_t, \sim_f, \wedge_t, \wedge_f, \vee_t, \vee_f, \rightarrow, \neg\}$ are presented below.

Definition 2 (L_{base})**.** The rules of L_{base} are of the form:

$$\frac{\alpha \quad \alpha\rightarrow\beta}{\beta}\ (\mathrm{mp}).$$

The axiom schemes of L_{base} are of the form:

(1) $\alpha\rightarrow(\beta\rightarrow\alpha)$,
(2) $(\alpha\rightarrow(\beta\rightarrow\gamma))\rightarrow((\alpha\rightarrow\beta)\rightarrow(\alpha\rightarrow\gamma))$,
(3) $(\alpha\wedge_t\beta)\rightarrow\alpha$,
(4) $(\alpha\wedge_t\beta)\rightarrow\beta$,
(5) $(\alpha\rightarrow\beta)\rightarrow((\alpha\rightarrow\gamma)\rightarrow(\alpha\rightarrow(\beta\wedge_t\gamma)))$,
(6) $\alpha\rightarrow(\alpha\vee_t\beta)$,
(7) $\beta\rightarrow(\alpha\vee_t\beta)$,
(8) $(\alpha\rightarrow\gamma)\rightarrow((\beta\rightarrow\gamma)\rightarrow((\alpha\vee_t\beta)\rightarrow\gamma))$,
(9) $\alpha\vee_t(\alpha\rightarrow\beta)$,
(10) $\alpha \leftrightarrow \sim_t\sim_t\alpha$,
(11) $\alpha \leftrightarrow \sim_f\sim_f\alpha$,
(12) $\sim_t\sim_f\alpha \leftrightarrow \sim_f\sim_t\alpha$,
(13) $\neg\sim_f\alpha \leftrightarrow \sim_f\neg\alpha$,
(14) $\neg\sim_t\alpha \leftrightarrow \sim_t\neg\alpha$,
(15) $\neg\sim_f\sim_t\alpha \leftrightarrow \sim_f\sim_t\neg\alpha$,
(16) $\sim_f\alpha \leftrightarrow \sim_t\sim_f\sim_t\alpha$,
(17) $\sim_t\alpha \leftrightarrow \sim_f\sim_t\sim_f\alpha$,
(18) $\sim_t(\alpha\wedge_t\beta) \leftrightarrow (\sim_t\alpha\vee_t\sim_t\beta)$,
(19) $\sim_t(\alpha\vee_t\beta) \leftrightarrow (\sim_t\alpha\wedge_t\sim_t\beta)$,
(20) $\sim_t(\alpha\wedge_f\beta) \leftrightarrow (\sim_t\alpha\wedge_f\sim_t\beta)$,
(21) $\sim_t(\alpha\vee_f\beta) \leftrightarrow (\sim_t\alpha\vee_f\sim_t\beta)$,
(22) $\sim_f(\alpha\wedge_f\beta) \leftrightarrow (\sim_f\alpha\vee_f\sim_f\beta)$,
(23) $\sim_f(\alpha\vee_f\beta) \leftrightarrow (\sim_f\alpha\wedge_f\sim_f\beta)$,
(24) $\sim_f(\alpha\wedge_t\beta) \leftrightarrow (\sim_f\alpha\wedge_t\sim_f\beta)$,
(25) $\sim_f(\alpha\vee_t\beta) \leftrightarrow (\sim_f\alpha\vee_t\sim_f\beta)$,
(26) $\sim_f\sim_t(\alpha\wedge_t\beta) \leftrightarrow (\sim_f\sim_t\alpha\vee_t\sim_f\sim_t\beta)$,
(27) $\sim_f\sim_t(\alpha\vee_t\beta) \leftrightarrow (\sim_f\sim_t\alpha\wedge_t\sim_f\sim_t\beta)$,
(28) $\sim_f\sim_t(\alpha\wedge_f\beta) \leftrightarrow (\sim_f\sim_t\alpha\wedge_f\sim_f\sim_t\beta)$,
(29) $\sim_f\sim_t(\alpha\vee_f\beta) \leftrightarrow (\sim_f\sim_t\alpha\vee_f\sim_f\sim_t\beta)$,
(30) $(\alpha\rightarrow\beta) \leftrightarrow (\neg\alpha\vee_t\beta)$,

(31) $\sim_t(\alpha\to\beta) \leftrightarrow (\sim_t\neg\alpha\wedge_t\sim_t\beta)$,
(32) $\sim_f(\alpha\to\beta) \leftrightarrow (\sim_f\alpha\to\sim_f\beta)$,
(33) $\sim_f\sim_t(\alpha\to\beta) \leftrightarrow (\sim_f\sim_t\neg\alpha\wedge_t\sim_f\sim_t\beta)$.

Definition 3 (L_B *and* L_T)**.** L_B is obtained from L_{base} by adding the axiom schemes of the form:

(34) $(\alpha\wedge_t\beta) \leftrightarrow (\alpha\vee_f\beta)$,
(35) $(\alpha\vee_t\beta) \leftrightarrow (\alpha\wedge_f\beta)$.

L_T is obtained from L_{base} by adding the axiom schemes of the form:

(36) $(\alpha\wedge_t\beta) \leftrightarrow (\alpha\wedge_f\beta)$,
(37) $(\alpha\vee_t\beta) \leftrightarrow (\alpha\vee_f\beta)$.

Since we can consider (cut-free) sequent calculi for both L_B and L_T similarly, we only discuss a sequent calculus for L_B and its first-order extension in the following sections. A cut-free sequent calculus for L_{base} has not been obtained yet. Such a calculus may be difficult to construct, see [14].

In the logical language of the sequent calculi L_{base}, L_B, and L_T, negation, conjunction, and disjunction come in two versions. In addition to the truth connectives $\sim_t$, $\wedge_t$, and $\vee_t$, there are also falsity connectives $\sim_f$, $\wedge_f$, and $\vee_f$. This split of the basic propositional vocabulary is induced by the truth order and the falsity order on the set of generalized truth values of *SIXTEEN*$_3$. The connectives $\wedge_t$ and $\vee_t$ ($\wedge_f$ and $\vee_f$) are interpreted as the lattice meet and lattice join (lattice join and lattice meet) of the truth order $\leq_t$ (the falsity order $\leq_f$), and the negations $\sim_t$ and $\sim_f$ are interpreted as certain truth order and falsity order inversions, respectively. Assignments of generalized truth values from *SIXTEEN*$_3$ to the propositional variables, i.e., functions from the set of propositional variables into the set **16**, the powerset of the powerset of the set of classical truth values, are thereby homomorphically extended to valuation functions v assigning generalized truth values to arbitrary formulas.

Definition 4. The binary entailment relations $\models_t$ and $\models_f$ on the set of formulas are defined by the following equivalences:

(1) $\alpha \models_t \beta$ iff $\forall v\ (v(\alpha) \leq_t v(\beta))$,
(2) $\alpha \models_f \beta$ iff $\forall v\ (v(\beta) \leq_f v(\alpha))$.

Note that $\models_t$ and $\models_f$ are distinct relations.

The basic vocabulary can naturally be extended by a truth implication and a falsity implication, interpreted as the residuum of the truth order and the falsity order, respectively. In [15], Odintsov showed that the relation $\models_t$ can be axiomatized if the truth implication $\to$ is added to the language. Classical negation $\neg$ can be defined by setting $\neg\alpha := \alpha \to \sim_t(p \to p)$, for some fixed atom p. Let $\top := p \to p$ (for some atom p) and $L^t = \{\alpha \mid \top \models_t \alpha\}$. Odintsov showed that $L^t = L_T \cap L_B$. The intersection of the theorems of L_T and those of L_B coincides with the set formulas $\models_t$-entailed by $\top$, where $\top$ is interpreted as the top-element of the truth order $\leq_t$.

3. Propositional case

3.1. Sequent calculus

Greek capital letters $\Gamma, \Delta, \ldots$ are used to represent finite (possibly empty) sets of formulas. An expression of the form $\Gamma \Rightarrow \Delta$ is called a *sequent*. An expression $L \vdash S$ (or $\vdash S$) is used to denote the fact that a sequent S is provable in a sequent calculus L.

Definition 5 (L_{16})**.** Let $\sim_b \in \{\sim_t\sim_f, \sim_f\sim_t\}$, $\sim_d \in \{\sim_t\sim_t, \sim_f\sim_f, \sim_b\sim_b\}$, and $\sim_e \in \{\sim_t, \sim_b\}$.

The initial sequents of L_{16} are of the form:

$$\alpha \Rightarrow \alpha \qquad \sim_f\sim_t\alpha \Rightarrow \sim_t\sim_f\alpha \qquad \sim_t\sim_f\alpha \Rightarrow \sim_f\sim_t\alpha.$$

The structural inference rules of L_{16} are of the form:

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \alpha, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (\text{cut}) \quad \frac{\Gamma \Rightarrow \Delta}{\alpha, \Gamma \Rightarrow \Delta}\ (\text{w-l}) \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \alpha}\ (\text{w-r}).$$

The logical inference rules of L_{16} are of the form:

$$\frac{\Gamma \Rightarrow \Sigma, \alpha \quad \beta, \Delta \Rightarrow \Pi}{\alpha\to\beta, \Gamma, \Delta \Rightarrow \Sigma, \Pi}\ (\to\text{l}) \qquad \frac{\alpha, \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha\to\beta}\ (\to\text{r})$$

$$\frac{\Gamma \Rightarrow \Delta, \alpha}{\neg\alpha, \Gamma \Rightarrow \Delta}\ (\neg\text{l}) \qquad \frac{\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\alpha}\ (\neg\text{r})$$

$$\frac{\alpha, \beta, \Gamma \Rightarrow \Delta}{\alpha\wedge_t\beta, \Gamma \Rightarrow \Delta}\ (\wedge_t\text{l}) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha \quad \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha\wedge_t\beta}\ (\wedge_t\text{r})$$

$$\frac{\alpha, \Gamma \Rightarrow \Delta \quad \beta, \Gamma \Rightarrow \Delta}{\alpha \vee_t \beta, \Gamma \Rightarrow \Delta}\ (\vee_t \mathrm{l}) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha, \beta}{\Gamma \Rightarrow \Delta, \alpha \vee_t \beta}\ (\vee_t \mathrm{r})$$

$$\frac{\alpha, \Gamma \Rightarrow \Delta \quad \beta, \Gamma \Rightarrow \Delta}{\alpha \wedge_f \beta, \Gamma \Rightarrow \Delta}\ (\wedge_f \mathrm{l}) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha, \beta}{\Gamma \Rightarrow \Delta, \alpha \wedge_f \beta}\ (\wedge_f \mathrm{r})$$

$$\frac{\alpha, \beta, \Gamma \Rightarrow \Delta}{\alpha \vee_f \beta, \Gamma \Rightarrow \Delta}\ (\vee_f \mathrm{l}) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha \quad \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \vee_f \beta}\ (\vee_f \mathrm{r})$$

$$\frac{\alpha, \Gamma \Rightarrow \Delta}{\sim_d \alpha, \Gamma \Rightarrow \Delta}\ (\sim_d \mathrm{l}) \quad \frac{\Gamma \Rightarrow \Delta, \alpha}{\Gamma \Rightarrow \Delta, \sim_d \alpha}\ (\sim_d \mathrm{r})$$

$$\frac{\sim_f \alpha, \Gamma \Rightarrow \Delta}{\sim_t \sim_f \sim_t \alpha, \Gamma \Rightarrow \Delta}\ (\sim_t \sim_f \sim_t \mathrm{l}) \quad \frac{\Gamma \Rightarrow \Delta, \sim_f \alpha}{\Gamma \Rightarrow \Delta, \sim_t \sim_f \sim_t \alpha}\ (\sim_t \sim_f \sim_t \mathrm{r})$$

$$\frac{\sim_t \alpha, \Gamma \Rightarrow \Delta}{\sim_f \sim_t \sim_f \alpha, \Gamma \Rightarrow \Delta}\ (\sim_f \sim_t \sim_f \mathrm{l}) \quad \frac{\Gamma \Rightarrow \Delta, \sim_t \alpha}{\Gamma \Rightarrow \Delta, \sim_f \sim_t \sim_f \alpha}\ (\sim_f \sim_t \sim_f \mathrm{r})$$

$$\frac{\neg \alpha, \Gamma \Rightarrow \Delta}{\sim_t \neg \sim_t \alpha, \Gamma \Rightarrow \Delta}\ (\sim_t \neg \sim_t \mathrm{l}) \quad \frac{\Gamma \Rightarrow \Delta, \neg \alpha}{\Gamma \Rightarrow \Delta, \sim_t \neg \sim_t \alpha}\ (\sim_t \neg \sim_t \mathrm{r})$$

$$\frac{\neg \alpha, \Gamma \Rightarrow \Delta}{\sim_f \neg \sim_f \alpha, \Gamma \Rightarrow \Delta}\ (\sim_f \neg \sim_f \mathrm{l}) \quad \frac{\Gamma \Rightarrow \Delta, \neg \alpha}{\Gamma \Rightarrow \Delta, \sim_f \neg \sim_f \alpha}\ (\sim_f \neg \sim_f \mathrm{r})$$

$$\frac{\sim_f \alpha, \Gamma \Rightarrow \Delta}{\sim_f \sim_t \sim_t \alpha, \Gamma \Rightarrow \Delta}\ (\sim_f \sim_t \sim_t \mathrm{l}) \quad \frac{\Gamma \Rightarrow \Delta, \sim_f \alpha}{\Gamma \Rightarrow \Delta, \sim_f \sim_t \sim_t \alpha}\ (\sim_f \sim_t \sim_t \mathrm{r})$$

$$\frac{\sim_t \alpha, \Gamma \Rightarrow \Delta}{\sim_t \sim_f \sim_f \alpha, \Gamma \Rightarrow \Delta}\ (\sim_t \sim_f \sim_f \mathrm{l}) \quad \frac{\Gamma \Rightarrow \Delta, \sim_t \alpha}{\Gamma \Rightarrow \Delta, \sim_t \sim_f \sim_f \alpha}\ (\sim_t \sim_f \sim_f \mathrm{r})$$

$$\frac{\neg \alpha, \Gamma \Rightarrow \Delta}{\neg \sim_t \sim_t \alpha, \Gamma \Rightarrow \Delta}\ (\neg \sim_t \sim_t \mathrm{l}) \quad \frac{\Gamma \Rightarrow \Delta, \neg \alpha}{\Gamma \Rightarrow \Delta, \neg \sim_t \sim_t \alpha}\ (\neg \sim_t \sim_t \mathrm{r})$$

$$\frac{\neg \alpha, \Gamma \Rightarrow \Delta}{\neg \sim_f \sim_f \alpha, \Gamma \Rightarrow \Delta}\ (\neg \sim_f \sim_f \mathrm{l}) \quad \frac{\Gamma \Rightarrow \Delta, \neg \alpha}{\Gamma \Rightarrow \Delta, \neg \sim_f \sim_f \alpha}\ (\neg \sim_f \sim_f \mathrm{r})$$

$$\frac{\sim_e \neg \alpha, \sim_e \beta, \Gamma \Rightarrow \Delta}{\sim_e (\alpha \rightarrow \beta), \Gamma \Rightarrow \Delta}\ (\sim_e \rightarrow \mathrm{l}) \qquad \frac{\Gamma \Rightarrow \Delta, \sim_e \neg \alpha \quad \Gamma \Rightarrow \Delta, \sim_e \beta}{\Gamma \Rightarrow \Delta, \sim_e (\alpha \rightarrow \beta)}\ (\sim_e \rightarrow \mathrm{r})$$

$$\frac{\Gamma \Rightarrow \Delta, \sim_e \alpha}{\sim_e \neg \alpha, \Gamma \Rightarrow \Delta}\ (\sim_e \neg \mathrm{l}) \quad \frac{\sim_e \alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \sim_e \neg \alpha}\ (\sim_e \neg \mathrm{r})$$

$$\frac{\sim_e \alpha, \Gamma \Rightarrow \Delta \quad \sim_e \beta, \Gamma \Rightarrow \Delta}{\sim_e (\alpha \wedge_t \beta), \Gamma \Rightarrow \Delta}\ (\sim_e \wedge_t \mathrm{l}) \qquad \frac{\Gamma \Rightarrow \Delta, \sim_e \alpha, \sim_e \beta}{\Gamma \Rightarrow \Delta, \sim_e (\alpha \wedge_t \beta)}\ (\sim_e \wedge_t \mathrm{r})$$

$$\frac{\sim_e \alpha, \sim_e \beta, \Gamma \Rightarrow \Delta}{\sim_e (\alpha \vee_t \beta), \Gamma \Rightarrow \Delta}\ (\sim_e \vee_t \mathrm{l}) \qquad \frac{\Gamma \Rightarrow \Delta, \sim_e \alpha \quad \Gamma \Rightarrow \Delta, \sim_e \beta}{\Gamma \Rightarrow \Delta, \sim_e (\alpha \vee_t \beta)}\ (\sim_e \vee_t \mathrm{r})$$

$$\frac{\sim_e \alpha, \sim_e \beta, \Gamma \Rightarrow \Delta}{\sim_e (\alpha \wedge_f \beta), \Gamma \Rightarrow \Delta}\ (\sim_e \wedge_f \mathrm{l}) \qquad \frac{\Gamma \Rightarrow \Delta, \sim_e \alpha \quad \Gamma \Rightarrow \Delta, \sim_e \beta}{\Gamma \Rightarrow \Delta, \sim_e (\alpha \wedge_f \beta)}\ (\sim_e \wedge_f \mathrm{r})$$

$$\frac{\sim_e \alpha, \Gamma \Rightarrow \Delta \quad \sim_e \beta, \Gamma \Rightarrow \Delta}{\sim_e (\alpha \vee_f \beta), \Gamma \Rightarrow \Delta}\ (\sim_e \vee_f \mathrm{l}) \qquad \frac{\Gamma \Rightarrow \Delta, \sim_e \alpha, \sim_e \beta}{\Gamma \Rightarrow \Delta, \sim_e (\alpha \vee_f \beta)}\ (\sim_e \vee_f \mathrm{r})$$

$$\frac{\Gamma \Rightarrow \Sigma, \sim_f \alpha \quad \sim_f \beta, \Delta \Rightarrow \Pi}{\sim_f (\alpha \rightarrow \beta), \Gamma, \Delta \Rightarrow \Sigma, \Pi}\ (\sim_f \rightarrow \mathrm{l}) \qquad \frac{\sim_f \alpha, \Gamma \Rightarrow \Delta, \sim_f \beta}{\Gamma \Rightarrow \Delta, \sim_f (\alpha \rightarrow \beta)}\ (\sim_f \rightarrow \mathrm{r})$$

$$\frac{\Gamma \Rightarrow \Delta, \sim_f \alpha}{\sim_f \neg \alpha, \Gamma \Rightarrow \Delta}\ (\sim_f \neg \mathrm{l}) \quad \frac{\sim_f \alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \sim_f \neg \alpha}\ (\sim_f \neg \mathrm{r})$$

$$\frac{\sim_f \alpha, \sim_f \beta, \Gamma \Rightarrow \Delta}{\sim_f (\alpha \wedge_t \beta), \Gamma \Rightarrow \Delta}\ (\sim_f \wedge_t \mathrm{l}) \qquad \frac{\Gamma \Rightarrow \Delta, \sim_f \alpha \quad \Gamma \Rightarrow \Delta, \sim_f \beta}{\Gamma \Rightarrow \Delta, \sim_f (\alpha \wedge_t \beta)}\ (\sim_f \wedge_t \mathrm{r})$$

$$\frac{\sim_f \alpha, \Gamma \Rightarrow \Delta \quad \sim_f \beta, \Gamma \Rightarrow \Delta}{\sim_f (\alpha \vee_t \beta), \Gamma \Rightarrow \Delta}\ (\sim_f \vee_t \mathrm{l}) \qquad \frac{\Gamma \Rightarrow \Delta, \sim_f \alpha, \sim_f \beta}{\Gamma \Rightarrow \Delta, \sim_f (\alpha \vee_t \beta)}\ (\sim_f \vee_t \mathrm{r})$$

$$\frac{{\sim_f}\alpha, \Gamma \Rightarrow \Delta \quad {\sim_f}\beta, \Gamma \Rightarrow \Delta}{{\sim_f}(\alpha \wedge_f \beta), \Gamma \Rightarrow \Delta}\ ({\sim_f}{\wedge_f}\mathrm{l}) \qquad \frac{\Gamma \Rightarrow \Delta, {\sim_f}\alpha, {\sim_f}\beta}{\Gamma \Rightarrow \Delta, {\sim_f}(\alpha \wedge_f \beta)}\ ({\sim_f}{\wedge_f}\mathrm{r})$$

$$\frac{{\sim_f}\alpha, {\sim_f}\beta, \Gamma \Rightarrow \Delta}{{\sim_f}(\alpha \vee_f \beta), \Gamma \Rightarrow \Delta}\ ({\sim_f}{\vee_f}\mathrm{l}) \qquad \frac{\Gamma \Rightarrow \Delta, {\sim_f}\alpha \quad \Gamma \Rightarrow \Delta, {\sim_f}\beta}{\Gamma \Rightarrow \Delta, {\sim_f}(\alpha \vee_f \beta)}\ ({\sim_f}{\vee_f}\mathrm{r}).$$

Note that the $\{\wedge_t, \vee_t, \sim_t\}$-fragment of L_{16} is a sequent calculus for Dunn and Belnap's four-valued logic [5,7] and that the inference rules $({\sim_t}{\wedge_f}\mathrm{l})$, $({\sim_t}{\wedge_f}\mathrm{r})$, $({\sim_t}{\vee_f}\mathrm{l})$ and $({\sim_t}{\vee_f}\mathrm{r})$ can be found in Arieli and Avron's bilattice logic [2], if $\wedge_f$ and $\vee_f$, respectively, are read as the (multiplicative) conjunction and disjunction connectives $*$ and $+$ used in [2]. Thus, L_{16} may be viewed as a natural extension and generalization of Dunn and Belnap's logic and Arieli and Avron's logic.

We also observe that a sequent calculus L^*_{16} for L_T is obtained from L_{16} by replacing the inference rules $\{(\wedge_f\mathrm{l}), (\wedge_f\mathrm{r}), (\vee_f\mathrm{l}), (\vee_f\mathrm{r})\}$ by the inference rules of the form:

$$\frac{\alpha, \beta, \Gamma \Rightarrow \Delta}{\alpha \wedge_f \beta, \Gamma \Rightarrow \Delta}\ (\wedge_f\mathrm{l}) \quad \frac{\Gamma \Rightarrow \Delta, \alpha \quad \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \wedge_f \beta}\ (\wedge_f\mathrm{r})$$

$$\frac{\alpha, \Gamma \Rightarrow \Delta \quad \beta, \Gamma \Rightarrow \Delta}{\alpha \vee_f \beta, \Gamma \Rightarrow \Delta}\ (\vee_f\mathrm{l}) \quad \frac{\Gamma \Rightarrow \Delta, \alpha, \beta}{\Gamma \Rightarrow \Delta, \alpha \vee_f \beta}\ (\vee_f\mathrm{r}).$$

Moreover, we note that L^*_{16} is an extension of the $\rightarrow$-free fragment of Arieli's and Avron's bilattice logic.

Proposition 6. *The following rules are derivable in cut-free* L_{16}*:*

$$\frac{{\sim_t}\alpha, \Gamma \Rightarrow \Delta}{\neg{\sim_t}\neg\alpha, \Gamma \Rightarrow \Delta}\ (\neg{\sim_t}\neg\mathrm{l}) \quad \frac{\Gamma \Rightarrow \Delta, {\sim_t}\alpha}{\Gamma \Rightarrow \Delta, \neg{\sim_t}\neg\alpha}\ (\neg{\sim_t}\neg\mathrm{r})$$

$$\frac{{\sim_f}\alpha, \Gamma \Rightarrow \Delta}{\neg{\sim_f}\neg\alpha, \Gamma \Rightarrow \Delta}\ (\neg{\sim_f}\neg\mathrm{l}) \quad \frac{\Gamma \Rightarrow \Delta, {\sim_f}\alpha}{\Gamma \Rightarrow \Delta, \neg{\sim_f}\neg\alpha}\ (\neg{\sim_f}\neg\mathrm{r})$$

$$\frac{{\sim_t}\alpha, \Gamma \Rightarrow \Delta}{{\sim_t}\neg\neg\alpha, \Gamma \Rightarrow \Delta}\ ({\sim_t}\neg\neg\mathrm{l}) \quad \frac{\Gamma \Rightarrow \Delta, {\sim_t}\alpha}{\Gamma \Rightarrow \Delta, {\sim_t}\neg\neg\alpha}\ ({\sim_t}\neg\neg\mathrm{r})$$

$$\frac{{\sim_f}\alpha, \Gamma \Rightarrow \Delta}{{\sim_f}\neg\neg\alpha, \Gamma \Rightarrow \Delta}\ ({\sim_f}\neg\neg\mathrm{l}) \quad \frac{\Gamma \Rightarrow \Delta, {\sim_f}\alpha}{\Gamma \Rightarrow \Delta, {\sim_f}\neg\neg\alpha}\ ({\sim_f}\neg\neg\mathrm{r}).$$

Proposition 7. *The following rules are admissible in cut-free* L_{16}*:*

$$\frac{{\sim_t}{\sim_f}\alpha, \Gamma \Rightarrow \Delta}{{\sim_f}{\sim_t}\alpha, \Gamma \Rightarrow \Delta}\ ({\sim_f}{\sim_t}\mathrm{l}) \quad \frac{\Gamma \Rightarrow \Delta, {\sim_t}{\sim_f}\alpha}{\Gamma \Rightarrow \Delta, {\sim_f}{\sim_t}\alpha}\ ({\sim_f}{\sim_t}\mathrm{r})$$

$$\frac{{\sim_f}{\sim_t}\alpha, \Gamma \Rightarrow \Delta}{{\sim_t}{\sim_f}\alpha, \Gamma \Rightarrow \Delta}\ ({\sim_t}{\sim_f}\mathrm{l}) \quad \frac{\Gamma \Rightarrow \Delta, {\sim_f}{\sim_t}\alpha}{\Gamma \Rightarrow \Delta, {\sim_t}{\sim_f}\alpha}\ ({\sim_t}{\sim_f}\mathrm{r})$$

$$\frac{{\sim_t}\neg\alpha, \Gamma \Rightarrow \Delta}{\neg{\sim_t}\alpha, \Gamma \Rightarrow \Delta}\ (\neg{\sim_t}\mathrm{l}) \quad \frac{\Gamma \Rightarrow \Delta, {\sim_t}\neg\alpha}{\Gamma \Rightarrow \Delta, \neg{\sim_t}\alpha}\ (\neg{\sim_t}\mathrm{r})$$

$$\frac{\neg{\sim_t}\alpha, \Gamma \Rightarrow \Delta}{{\sim_t}\neg\alpha, \Gamma \Rightarrow \Delta}\ ({\sim_t}\neg\mathrm{l}) \quad \frac{\Gamma \Rightarrow \Delta, \neg{\sim_t}\alpha}{\Gamma \Rightarrow \Delta, {\sim_t}\neg\alpha}\ ({\sim_t}\neg\mathrm{r})$$

$$\frac{\neg{\sim_f}\alpha, \Gamma \Rightarrow \Delta}{{\sim_f}\neg\alpha, \Gamma \Rightarrow \Delta}\ ({\sim_f}\neg\mathrm{l}) \quad \frac{\Gamma \Rightarrow \Delta, \neg{\sim_f}\alpha}{\Gamma \Rightarrow \Delta, {\sim_f}\neg\alpha}\ ({\sim_f}\neg\mathrm{r})$$

$$\frac{{\sim_f}\neg\alpha, \Gamma \Rightarrow \Delta}{\neg{\sim_f}\alpha, \Gamma \Rightarrow \Delta}\ (\neg{\sim_f}\mathrm{l}) \quad \frac{\Gamma \Rightarrow \Delta, {\sim_f}\neg\alpha}{\Gamma \Rightarrow \Delta, \neg{\sim_f}\alpha}\ (\neg{\sim_f}\mathrm{r})$$

$$\frac{{\sim_b}\neg\alpha, \Gamma \Rightarrow \Delta}{\neg{\sim_b}\alpha, \Gamma \Rightarrow \Delta}\ (\neg{\sim_b}\mathrm{l}) \quad \frac{\Gamma \Rightarrow \Delta, {\sim_b}\neg\alpha}{\Gamma \Rightarrow \Delta, \neg{\sim_b}\alpha}\ (\neg{\sim_b}\mathrm{r})$$

$$\frac{\neg{\sim_b}\alpha, \Gamma \Rightarrow \Delta}{{\sim_b}\neg\alpha, \Gamma \Rightarrow \Delta}\ ({\sim_b}\neg\mathrm{l}) \quad \frac{\Gamma \Rightarrow \Delta, \neg{\sim_b}\alpha}{\Gamma \Rightarrow \Delta, {\sim_b}\neg\alpha}\ ({\sim_b}\neg\mathrm{r}).$$

Proof. We show only the claim for $({\sim_f}{\sim_t}\mathrm{l})$ and $(\neg{\sim_t}\mathrm{r})$.

• $({\sim_f}{\sim_t}\mathrm{l})$: We show that the rule $({\sim_f}{\sim_t}\mathrm{l})$ is admissible in cut-free L_{16}, i.e., we show that $L_{16} - (\mathrm{cut}) \vdash {\sim_t}{\sim_f}\alpha, \Gamma \Rightarrow \Delta$ implies $L_{16} - (\mathrm{cut}) \vdash {\sim_f}{\sim_t}\alpha, \Gamma \Rightarrow \Delta$. This is proved by induction on a cut-free proof P of ${\sim_t}{\sim_f}\alpha, \Gamma \Rightarrow \Delta$. We distinguish the cases according to the last inference of P. We show some cases.

Case $({\sim_t}{\sim_f}p \Rightarrow {\sim_t}{\sim_f}p)$: P is of the form: ${\sim_t}{\sim_f}p \Rightarrow {\sim_t}{\sim_f}p$, where p is a propositional variable. Then, ${\sim_f}{\sim_t}p \Rightarrow {\sim_t}{\sim_f}p$ is an initial sequent of L_{16}.

Case $({\sim_t}{\sim_f}{\sim_t}\mathrm{l})$: The last inference of P is of the form:

$$\frac{{\sim_f}\beta, \Gamma \Rightarrow \Delta}{{\sim_t}{\sim_f}{\sim_t}\beta, \Gamma \Rightarrow \Delta}\ ({\sim_t}{\sim_f}{\sim_t}\mathrm{l})$$

where $\alpha \equiv {\sim_t}\beta$. By the hypothesis, we have the required fact:

$$\frac{{\sim_f}\beta, \Gamma \Rightarrow \Delta}{{\sim_f}{\sim_t}{\sim_t}\beta, \Gamma \Rightarrow \Delta}\ ({\sim_f}{\sim_t}{\sim_t}\mathrm{l}).$$

Case $({\sim_b}\wedge_t\mathrm{l})$: The last inference of P is of the form:

$$\frac{{\sim_t}{\sim_f}\alpha', \Gamma \Rightarrow \Delta \quad {\sim_t}{\sim_f}\alpha'', \Gamma \Rightarrow \Delta}{{\sim_t}{\sim_f}(\alpha'\wedge_t\alpha''), \Gamma \Rightarrow \Delta}\ ({\sim_b}\wedge_t\mathrm{l})$$

where $\alpha \equiv \alpha'\wedge_t\alpha''$. By the induction hypothesis, we have $\mathrm{L}_{16} - (\mathrm{cut}) \vdash {\sim_f}{\sim_t}\alpha', \Gamma \Rightarrow \Delta$ and $\mathrm{L}_{16} - (\mathrm{cut}) \vdash {\sim_f}{\sim_t}\alpha'', \Gamma \Rightarrow \Delta$. Thus, we obtain the required fact:

$$\frac{{\sim_f}{\sim_t}\alpha', \Gamma \Rightarrow \Delta \quad {\sim_f}{\sim_t}\alpha'', \Gamma \Rightarrow \Delta}{{\sim_f}{\sim_t}(\alpha'\wedge_t\alpha''), \Gamma \Rightarrow \Delta}\ ({\sim_b}\wedge_t\mathrm{l}).$$

• $(\neg{\sim_t}\mathrm{r})$: We show that the rule $(\neg{\sim_t}\mathrm{r})$ is admissible in cut-free L_{16}, i.e., we show that $\mathrm{L}_{16} - (\mathrm{cut}) \vdash \Gamma \Rightarrow \Delta, {\sim_t}\neg\alpha$ implies $\mathrm{L}_{16} - (\mathrm{cut}) \vdash \Gamma \Rightarrow \Delta, \neg{\sim_t}\alpha$. This is proved by induction on a cut-free proof P of $\Gamma \Rightarrow \Delta, {\sim_t}\neg\alpha$. We distinguish the cases according to the last inference of P. We show some cases.

Case $({\sim_t}\neg{\sim_t}\mathrm{r})$: The last inference of P is of the form:

$$\frac{\Gamma \Rightarrow \Delta, \neg\beta}{\Gamma \Rightarrow \Delta, {\sim_t}\neg{\sim_t}\beta}\ ({\sim_t}\neg{\sim_t}\mathrm{r})$$

where $\alpha \equiv {\sim_t}\beta$. By the hypothesis, we obtain the required fact:

$$\frac{\Gamma \Rightarrow \Delta, \neg\beta}{\Gamma \Rightarrow \Delta, \neg{\sim_t}{\sim_t}\beta}\ (\neg{\sim_t}{\sim_t}\mathrm{r}).$$

Case $({\sim_t}\neg\mathrm{r})$: The last inference of P is of the form:

$$\frac{{\sim_t}\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, {\sim_t}\neg\alpha}\ ({\sim_t}\neg\mathrm{r}).$$

By the hypothesis, we obtain the required fact:

$$\frac{{\sim_t}\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg{\sim_t}\alpha}\ (\neg\mathrm{r}). \quad \square$$

The rules presented in Proposition 7 are quite natural with regard to obtaining a sequent system for Odintsov's axiom system L_B and, in fact, the sequent system GL_B for L_B defined in [14] differs from L_{16} inter alia in that it comprises these admissible rules. These rules are not adopted in the definition of L_{16}, because they are not suited for applying Maehara's method, see Section 3.3.

Proposition 8 (*Equivalence between* L_{16} *and* GL_B)**.** *Let* Γ, Δ *be finite sets of formulas from the common object language of* L_{16} *and* GL_B. *Then* $\mathrm{L}_{16} \vdash \Gamma \Rightarrow \Delta$ *iff* $\mathrm{GL}_B \vdash \Gamma \Rightarrow \Delta$.

Proof. All axioms of GL_B are provable in L_{16}, and all rules of GL_B are either primitive or admissible rules of L_{16}. Conversely, all axioms of L_{16} are provable in GL_B, and all rules of L_{16} are either primitive or admissible rules of GL_B. $\square$

Theorem 9 (*Equivalence between* L_{16} *and* L_B)**.** $\mathrm{L}_{16} \vdash\ \Rightarrow \alpha$ *iff* $\mathrm{L}_B \vdash \alpha$.

Proof. By Proposition 8 and Theorem 5.8 in [14], showing the equivalence between GL_B and L_B. $\square$

3.2. Semantics

The semantics which we will extend to a semantics for first-order trilattice logics makes use of Odintsov's co-ordinate valuations [15]. These valuations are defined on the basis of Odintsov's matrix representation of the following operations on **16**: $\sqcap_t$ (lattice meet of the truth order), $\sqcup_t$ (lattice join of the truth order), $-_t$ (truth order inversion), $\sqcap_f$ (lattice meet of the falsity order), $\sqcup_f$ (lattice join of the falsity order), $-_f$ (falsity order inversion), $\sqsupseteq_t$ (residuum of $\sqcap_t$ with respect to the truth order), see Section 2.1 and [14,15,19,20,22]. Every element of **16** is a subset of the powerset of the set of classical truth values T (true) and F (*false*), i.e., it is a subset of $\{\mathbf{N} = \emptyset, \mathbf{T} = \{T\}, \mathbf{F} = \{F\}, \mathbf{B} = \{T, F\}\}$. Therefore, every element x of **16** can be represented as a 2×2-matrix of values of characteristic functions:

$$\begin{vmatrix} n & f \\ t & b \end{vmatrix}$$

where each element of the matrix is an element of the set $\{0, 1\}$ and the following equivalences hold:

$$n = 1 \text{ iff } \mathbf{N} \in x; \quad f = 1 \text{ iff } \mathbf{F} \in x; \quad t = 1 \text{ iff } \mathbf{T} \in x; \quad b = 1 \text{ iff } \mathbf{B} \in x.$$

Proposition 10 (*Odintsov [15]*). *Let $\wedge$ and $\vee$ be the classical truth functions of conjunction and disjunction, let $x, y \in$ **16**, let*

$$x = \begin{vmatrix} n & f \\ t & b \end{vmatrix} \quad \text{and let } y = \begin{vmatrix} y & f' \\ t' & b' \end{vmatrix}.$$

Then the following equations hold:

$$\begin{vmatrix} n & f \\ t & b \end{vmatrix} \sqcap_t \begin{vmatrix} n' & f' \\ t' & b' \end{vmatrix} = \begin{vmatrix} n \vee n' & f \vee f' \\ t \wedge t' & b \wedge b' \end{vmatrix}$$

$$\begin{vmatrix} n & f \\ t & b \end{vmatrix} \sqcup_t \begin{vmatrix} n' & f' \\ t' & b' \end{vmatrix} = \begin{vmatrix} n \wedge n' & f \wedge f' \\ t \vee t' & b \vee b' \end{vmatrix}$$

$$-_t \begin{vmatrix} n & f \\ t & b \end{vmatrix} = \begin{vmatrix} t & b \\ n & f \end{vmatrix}$$

$$\begin{vmatrix} n & f \\ t & b \end{vmatrix} \sqcap_f \begin{vmatrix} n' & f' \\ t' & b' \end{vmatrix} = \begin{vmatrix} n \wedge n' & f \vee f' \\ t \wedge t' & b \vee b' \end{vmatrix}$$

$$\begin{vmatrix} n & f \\ t & b \end{vmatrix} \sqcup_f \begin{vmatrix} n' & f' \\ t' & b' \end{vmatrix} = \begin{vmatrix} n \vee n' & f \wedge f' \\ t \vee t' & b \wedge b' \end{vmatrix}$$

$$-_f \begin{vmatrix} n & f \\ t & b \end{vmatrix} = \begin{vmatrix} f & n \\ b & t \end{vmatrix}.$$

Moreover,

$$\begin{vmatrix} n & f \\ t & b \end{vmatrix} \sqsupset_t \begin{vmatrix} n' & f' \\ t' & b' \end{vmatrix} = \begin{vmatrix} \neg n \wedge n' & \neg f \wedge f' \\ t \rightarrow t' & b \rightarrow b' \end{vmatrix}$$

where $\rightarrow$ and $\neg$ denote the truth functions of Boolean implication and Boolean negation, respectively.

In view of the matrix presentation of elements of **16**, every assignment v from the set of propositional variables into **16** can be associated with co-ordinate valuations v_t, v_f, v_n, and v_b which are classical valuations from the set of propositional variables into $\{0, 1\}$. The co-ordinate valuations are defined by the following equivalence:

$$v_c(p) = 1 \quad \text{iff } \mathbf{C} \in v(p), \ c \in \{n, f, t, b\}.$$

In this section, we shall introduce valuation functions v^n, v^f, v^t, v^b and use them to prove strong completeness and cut-elimination for L_{16}. Moreover, we explain how these mappings are related to Odintsov's co-ordinate valuations.

Let p be a fixed propositional variable. Suppose Γ is a set $\{\alpha_1, \ldots, \alpha_m\}$ ($m \geq 0$) of formulas. Then Γ^* is defined as $\alpha_1 \vee_t \cdots \vee_t \alpha_m$ if $m \geq 1$, and $\neg(p \rightarrow p)$ if $m = 0$. Also Γ_* is defined as $\alpha_1 \wedge_t \cdots \wedge_t \alpha_m$ if $m \geq 1$, and $p \rightarrow p$ if $m = 0$. In the following discussion, the commutativity of $\wedge_t$ or $\vee_t$ is assumed. We have the following fact: for any formulas $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n$, the sequent $\alpha_1, \ldots, \alpha_m \Rightarrow \beta_1, \ldots, \beta_n$ is provable in L_{16} if and only if so is $\alpha_1 \wedge_t \cdots \wedge_t \alpha_m \Rightarrow \beta_1 \vee_t \cdots \vee_t \beta_n$.

Definition 11. The valuations v^n, v^t, v^f and v^b are mappings from the set of all propositional variables to the set $\{t, f\}$. The valuations v^n, v^t, v^f and v^b are extended to mappings from the set of all formulas to $\{t, f\}$ by the following clauses. For any $e \in \{t, b\}, g \in \{f, n\}$:

(1) $v^g(\alpha \rightarrow \beta) = t$ iff $v^g(\alpha) = f$ or $v^g(\beta) = t$,
(2) $v^g(\neg\alpha) = t$ iff $v^g(\alpha) = f$,
(3) $v^g(\alpha \wedge_t \beta) = t$ iff $v^g(\alpha) = t$ and $v^g(\beta) = t$,
(4) $v^g(\alpha \vee_t \beta) = t$ iff $v^g(\alpha) = t$ or $v^g(\beta) = t$,
(5) $v^g(\alpha \wedge_f \beta) = t$ iff $v^g(\alpha) = t$ or $v^g(\beta) = t$,
(6) $v^g(\alpha \vee_f \beta) = t$ iff $v^g(\alpha) = t$ and $v^g(\beta) = t$,
(7) $v^n({\sim_t}\alpha) = t$ iff $v^t(\alpha) = t$,
(8) $v^n({\sim_f}\alpha) = t$ iff $v^f(\alpha) = t$,
(9) $v^n({\sim_b}\alpha) = t$ iff $v^b(\alpha) = t$,
(10) $v^e(\alpha \rightarrow \beta) = t$ iff $v^e(\alpha) = f$ and $v^e(\beta) = t$,
(11) $v^e(\neg\alpha) = t$ iff $v^e(\alpha) = f$,
(12) $v^e(\alpha \wedge_t \beta) = t$ iff $v^e(\alpha) = t$ or $v^e(\beta) = t$,
(13) $v^e(\alpha \vee_t \beta) = t$ iff $v^e(\alpha) = t$ and $v^e(\beta) = t$,
(14) $v^e(\alpha \wedge_f \beta) = t$ iff $v^e(\alpha) = t$ and $v^e(\beta) = t$,
(15) $v^e(\alpha \vee_f \beta) = t$ iff $v^e(\alpha) = t$ or $v^e(\beta) = t$,
(16) $v^t({\sim_t}\alpha) = t$ iff $v^n(\alpha) = t$,
(17) $v^t({\sim_f}\alpha) = t$ iff $v^b(\alpha) = t$,
(18) $v^t({\sim_b}\alpha) = t$ iff $v^f(\alpha) = t$,

(19) $v^f(\sim_t\alpha) = t$ iff $v^b(\alpha) = t$,
(20) $v^f(\sim_f\alpha) = t$ iff $v^n(\alpha) = t$,
(21) $v^f(\sim_b\alpha) = t$ iff $v^t(\alpha) = t$,
(22) $v^b(\sim_t\alpha) = t$ iff $v^f(\alpha) = t$,
(23) $v^b(\sim_f\alpha) = t$ iff $v^t(\alpha) = t$,
(24) $v^b(\sim_b\alpha) = t$ iff $v^n(\alpha) = t$.

A formula α is called a *tautology* if $v^n(\alpha) = t$ holds for any valuations v^n, v^t, v^f and v^b. A sequent of the form $\Gamma \Rightarrow \Delta$ is called a tautology if so is the formula $\Gamma_* \rightarrow \Delta^*$.

Note that the valuation v^n behaves classically with respect to the classical connectives $\wedge_t$, $\vee_t$, $\neg$, and $\rightarrow$. This may be seen as a justification for defining the notion of a tautology with respect to v^n. Moreover, it is noted that the following conditions hold: For any $* \in \{n, t, f, b\}$,

(1) $v^*(\alpha\wedge_t\beta) = v^*(\alpha\vee_f\beta)$,
(2) $v^*(\alpha\vee_t\beta) = v^*(\alpha\wedge_f\beta)$,
(3) $v^t(\sim_f\alpha) = v^f(\sim_t\alpha)$,
(4) $v^t(\alpha) = v^n(\sim_t\alpha) = v^f(\sim_b\alpha) = v^b(\sim_f\alpha)$,
(5) $v^f(\alpha) = v^n(\sim_f\alpha) = v^t(\sim_b\alpha) = v^b(\sim_t\alpha)$,
(6) $v^b(\alpha) = v^n(\sim_b\alpha) = v^t(\sim_f\alpha) = v^f(\sim_t\alpha)$.

Theorem 12 (*Soundness for* L_{16})**.** *For any sequent S, if* $\mathrm{L}_{16} \vdash S$*, then S is a tautology.*

3.3. Completeness and cut-elimination

In the following, we prove the (strong) completeness and cut-elimination theorems for L_{16} by using the method by Maehara presented, for instance, in [16].

Definition 13. Let $\sim_d \in \{\sim_t\sim_t, \sim_f\sim_f, \sim_b\sim_b\}$ and $\sim_e \in \{\sim_t, \sim_b\}$.

A *decomposition* of a sequent S is defined as having the form S' or S'; S'' by

(1) $\Gamma \Rightarrow \Delta, \alpha$; $\beta, \Gamma \Rightarrow \Delta$ is a decomposition of $\alpha\rightarrow\beta, \Gamma \Rightarrow \Delta$,
(2) $\alpha, \Gamma \Rightarrow \Delta, \beta$ is a decomposition of $\Gamma \Rightarrow \Delta, \alpha\rightarrow\beta$,
(3) $\Gamma \Rightarrow \Delta, \alpha$ is a decomposition of $\neg\alpha, \Gamma \Rightarrow \Delta$,
(4) $\alpha, \Gamma \Rightarrow \Delta$ is a decomposition of $\Gamma \Rightarrow \Delta, \neg\alpha$,
(5) $\alpha, \beta, \Gamma \Rightarrow \Delta$ is a decomposition of $\alpha\wedge_t\beta, \Gamma \Rightarrow \Delta$,
(6) $\Gamma \Rightarrow \Delta, \alpha$; $\Gamma \Rightarrow \Delta, \beta$ is a decomposition of $\Gamma \Rightarrow \Delta, \alpha\wedge_t\beta$,
(7) $\alpha, \Gamma \Rightarrow \Delta$; $\beta, \Gamma \Rightarrow \Delta$ is a decomposition of $\alpha\vee_t\beta, \Gamma \Rightarrow \Delta$,
(8) $\Gamma \Rightarrow \Delta, \alpha, \beta$ is a decomposition of $\Gamma \Rightarrow \Delta, \alpha\vee_t\beta$,
(9) $\alpha, \Gamma \Rightarrow \Delta$; $\beta, \Gamma \Rightarrow \Delta$ is a decomposition of $\alpha\wedge_f\beta, \Gamma \Rightarrow \Delta$,
(10) $\Gamma \Rightarrow \Delta, \alpha, \beta$ is a decomposition of $\Gamma \Rightarrow \Delta, \alpha\wedge_f\beta$,
(11) $\alpha, \beta, \Gamma \Rightarrow \Delta$ is a decomposition of $\alpha\vee_f\beta, \Gamma \Rightarrow \Delta$,
(12) $\Gamma \Rightarrow \Delta, \alpha$; $\Gamma \Rightarrow \Delta, \beta$ is a decomposition of $\Gamma \Rightarrow \Delta, \alpha\vee_f\beta$,
(13) $\alpha, \Gamma \Rightarrow \Delta$ is a decomposition of $\sim_d\alpha, \Gamma \Rightarrow \Delta$,
(14) $\Gamma \Rightarrow \Delta, \alpha$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim_d\alpha$,
(15) $\sim_f\alpha, \Gamma \Rightarrow \Delta$ is a decomposition of $\sim_t\sim_f\sim_t\alpha, \Gamma \Rightarrow \Delta$,
(16) $\Gamma \Rightarrow \Delta, \sim_f\alpha$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim_t\sim_f\sim_t\alpha$,
(17) $\sim_t\alpha, \Gamma \Rightarrow \Delta$ is a decomposition of $\sim_f\sim_t\sim_f\alpha, \Gamma \Rightarrow \Delta$,
(18) $\Gamma \Rightarrow \Delta, \sim_t\alpha$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim_f\sim_t\sim_f\alpha$,
(19) $\neg\alpha, \Gamma \Rightarrow \Delta$ is a decomposition of $\sim_t\neg\sim_t\alpha, \Gamma \Rightarrow \Delta$,
(20) $\Gamma \Rightarrow \Delta, \neg\alpha$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim_t\neg\sim_t\alpha$,
(21) $\neg\alpha, \Gamma \Rightarrow \Delta$ is a decomposition of $\sim_f\neg\sim_f\alpha, \Gamma \Rightarrow \Delta$,
(22) $\Gamma \Rightarrow \Delta, \neg\alpha$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim_f\neg\sim_f\alpha$,
(23) $\sim_f\alpha, \Gamma \Rightarrow \Delta$ is a decomposition of $\sim_f\sim_t\sim_t\alpha, \Gamma \Rightarrow \Delta$,
(24) $\Gamma \Rightarrow \Delta, \sim_f\alpha$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim_f\sim_t\sim_t\alpha$,
(25) $\sim_t\alpha, \Gamma \Rightarrow \Delta$ is a decomposition of $\sim_t\sim_f\sim_f\alpha, \Gamma \Rightarrow \Delta$,
(26) $\Gamma \Rightarrow \Delta, \sim_t\alpha$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim_t\sim_f\sim_f\alpha$,
(27) $\neg\alpha, \Gamma \Rightarrow \Delta$ is a decomposition of $\neg\sim_t\sim_t\alpha, \Gamma \Rightarrow \Delta$,
(28) $\Gamma \Rightarrow \Delta, \neg\alpha$ is a decomposition of $\Gamma \Rightarrow \Delta, \neg\sim_t\sim_t\alpha$,
(29) $\neg\alpha, \Gamma \Rightarrow \Delta$ is a decomposition of $\neg\sim_f\sim_f\alpha, \Gamma \Rightarrow \Delta$,
(30) $\Gamma \Rightarrow \Delta, \neg\alpha$ is a decomposition of $\Gamma \Rightarrow \Delta, \neg\sim_f\sim_f\alpha$,
(31) $\sim_e\neg\alpha, \sim_e\beta, \Gamma \Rightarrow \Delta$ is a decomposition of $\sim_e(\alpha\rightarrow\beta), \Gamma \Rightarrow \Delta$,
(32) $\Gamma \Rightarrow \Delta, \sim_e\neg\alpha$; $\Gamma \Rightarrow \Delta, \sim_e\beta$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim_e(\alpha\rightarrow\beta)$,
(33) $\Gamma \Rightarrow \Delta, \sim_e\alpha$ is a decomposition of $\sim_e\neg\alpha, \Gamma \Rightarrow \Delta$,

(34) $\sim_e\alpha, \Gamma \Rightarrow \Delta$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim_e\neg\alpha$,
(35) $\sim_e\alpha, \Gamma \Rightarrow \Delta$; $\sim_e\beta, \Gamma \Rightarrow \Delta$ is a decomposition of $\sim_e(\alpha\wedge_t\beta), \Gamma \Rightarrow \Delta$,
(36) $\Gamma \Rightarrow \Delta, \sim_e\alpha, \sim_e\beta$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim_e(\alpha\wedge_t\beta)$,
(37) $\sim_e\alpha, \sim_e\beta, \Gamma \Rightarrow \Delta$ is a decomposition of $\sim_e(\alpha\vee_t\beta), \Gamma \Rightarrow \Delta$,
(38) $\Gamma \Rightarrow \Delta, \sim_e\alpha$; $\Gamma \Rightarrow \Delta, \sim_e\beta$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim_e(\alpha\vee_t\beta)$,
(39) $\sim_e\alpha, \sim_e\beta, \Gamma \Rightarrow \Delta$ is a decomposition of $\sim_e(\alpha\wedge_f\beta), \Gamma \Rightarrow \Delta$,
(40) $\Gamma \Rightarrow \Delta, \sim_e\alpha$; $\Gamma \Rightarrow \Delta, \sim_e\beta$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim_e(\alpha\wedge_f\beta)$,
(41) $\sim_e\alpha, \Gamma \Rightarrow \Delta$; $\sim_e\beta, \Gamma \Rightarrow \Delta$ is a decomposition of $\sim_e(\alpha\vee_f\beta), \Gamma \Rightarrow \Delta$,
(42) $\Gamma \Rightarrow \Delta, \sim_e\alpha, \sim_e\beta$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim_e(\alpha\vee_f\beta)$,
(43) $\Gamma \Rightarrow \Delta, \sim_f\alpha$; $\sim_f\beta, \Gamma \Rightarrow \Delta$ is a decomposition of $\sim_f(\alpha\rightarrow\beta), \Gamma \Rightarrow \Delta$,
(44) $\sim_f\alpha, \Gamma \Rightarrow \Delta, \sim_f\beta$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim_f(\alpha\rightarrow\beta)$,
(45) $\Gamma \Rightarrow \Delta, \sim_f\alpha$ is a decomposition of $\sim_f(\neg\alpha), \Gamma \Rightarrow \Delta$,
(46) $\sim_f\alpha, \Gamma \Rightarrow \Delta$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim_f(\neg\alpha)$,
(47) $\sim_f\alpha, \sim_f\beta, \Gamma \Rightarrow \Delta$ is a decomposition of $\sim_f(\alpha\wedge_t\beta), \Gamma \Rightarrow \Delta$,
(48) $\Gamma \Rightarrow \Delta, \sim_f\alpha$; $\Gamma \Rightarrow \Delta, \sim_f\beta$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim_f(\alpha\wedge_t\beta)$,
(49) $\sim_f\alpha, \Gamma \Rightarrow \Delta$; $\sim_f\beta, \Gamma \Rightarrow \Delta$ is a decomposition of $\sim_f(\alpha\vee_t\beta), \Gamma \Rightarrow \Delta$,
(50) $\Gamma \Rightarrow \Delta, \sim_f\alpha, \sim_f\beta$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim_f(\alpha\vee_t\beta)$,
(51) $\sim_f\alpha, \Gamma \Rightarrow \Delta$; $\sim_f\beta, \Gamma \Rightarrow \Delta$ is a decomposition of $\sim_f(\alpha\wedge_f\beta), \Gamma \Rightarrow \Delta$,
(52) $\Gamma \Rightarrow \Delta, \sim_f\alpha, \sim_f\beta$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim_f(\alpha\wedge_f\beta)$,
(53) $\sim_f\alpha, \sim_f\beta, \Gamma \Rightarrow \Delta$ is a decomposition of $\sim_f(\alpha\vee_f\beta), \Gamma \Rightarrow \Delta$,
(54) $\Gamma \Rightarrow \Delta, \sim_f\alpha$; $\Gamma \Rightarrow \Delta, \sim_f\beta$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim_f(\alpha\vee_f\beta)$.

Note that the clauses in the definition of the decompositions just correspond to the logical inference rules of L_{16}.

Definition 14. A *decomposition tree of* a sequent S is a tree which expresses a process of some repeated decomposition of S. In other words, a decomposition tree corresponds to a bottom up proof search tree. A *complete decomposition tree of S* is a decomposition tree of a sequent S in which all the formulas occurring in all the leaves of the tree are of one of the following forms: p, $\sim_t p$, $\sim_f p$ and $\sim_b p$.

Lemma 15. *For any sequent S, there is a complete decomposition tree of S, i.e., every repeated decomposition process terminates.*

Proof. By the definition of decomposition, S_1 and S_2 consist only of some subformulas or negated subformulas of a formula in S. □

Note that if the corresponding decomposition rules of the admissible inference rules displayed in Proposition 7 are adopted, then some repeated decomposition processes do not terminate. This is the reason why we do not use the inference rules in Proposition 7 in the definition of L_{16}.

Lemma 16. *Let S_1 or S_1 ; S_2 be a decomposition of S. If S is a tautology, then so are S_1 and S_2.*

Proof. We show some cases.

• (15): Suppose that $\sim_t\sim_f\sim_t\alpha\wedge_t\Gamma_*\rightarrow\Delta^*$ is a tautology. We show that the sequent $\sim_f\alpha\wedge_t\Gamma_*\rightarrow\Delta^*$ is a tautology. Suppose (1) $v^n(\sim_f\alpha\wedge_t\Gamma_*) = t$. We show $v^n(\Delta^*) = t$. By (1), we have (2) $v^n(\sim_f\alpha) = v^f(\alpha) = t$ and (3) $v^n(\Gamma_*) = t$. By (2), we obtain (4) $v^n(\sim_t\sim_f\sim_t\alpha) = v^n(\sim_t\sim_b\alpha) = v^t(\sim_b\alpha) = v^f(\alpha) = t$. On the other hand, we have (5) $v^n(\sim_t\sim_f\sim_t\alpha\wedge_t\Gamma_*\rightarrow\Delta^*) = t$ by the hypothesis. By (5), (4), and (3), we obtain $v^n(\Delta^*) = t$.

• (21): Suppose that $\sim_f\neg\sim_f\alpha\wedge_t\Gamma_*\rightarrow\Delta^*$ is a tautology. We show that the sequent $\neg\alpha\wedge_t\Gamma_*\rightarrow\Delta^*$ is a tautology. Suppose $v^n(\neg\alpha\wedge_t\Gamma_*) = t$, i.e., (1) $v^n(\neg\alpha) = t$ and (2) $v^n(\Gamma_*) = t$. Then, we show $v^n(\Delta^*) = t$. We have (3) $v^n(\sim_f\neg\sim_f\alpha) = v^f(\neg\sim_f\alpha) = t$ iff $v^f(\sim_f\alpha) = f$ iff $v^n(\alpha) = f$ iff $v^n(\neg\alpha) = t$. Thus, we obtain (4) $v^n(\sim_f\neg\sim_f\alpha) = t$ by (3) and (1). On the other hand, we have (5) $v^n(\sim_f\neg\sim_f\alpha\wedge_t\Gamma_*\rightarrow\Delta^*) = t$ by the hypothesis. Thus, we obtain $v^n(\Delta^*) = t$ by (5), (4), and (2).

• (32): We show only the case $\sim_e = \sim_t$. Suppose that $\Gamma_*\rightarrow\Delta^*\vee_t\sim_t(\alpha\rightarrow\beta)$ is a tautology. First, we show that $\Gamma_* \rightarrow \Delta^* \vee_t\sim_t\neg\alpha$ is a tautology. Suppose that (1) $v^n(\Gamma_*) = t$. We show $v^n(\Delta^*\vee_t\sim_t\neg\alpha) = t$. If $v^n(\sim_t\neg\alpha) = t$, then $v^n(\Delta^*\vee_t\sim_t\neg\alpha) = t$. Thus, suppose $v^n(\sim_t\neg\alpha) = v^t(\neg\alpha) = f$, i.e., $v^t(\alpha) = t$. Then, $v^t(\alpha\rightarrow\beta) = f$, since $v^t(\alpha\rightarrow\beta) = f$ iff $v^t(\alpha) = t$ or $v^t(\beta) = f$. Hence, we have (2) $v^n(\sim_t(\alpha\rightarrow\beta)) = f$. On the other hand, we have (3) $v^n(\Gamma_*\rightarrow\Delta^*\vee_t\sim_t(\alpha\rightarrow\beta)) = t$ by the hypothesis. Thus, we obtain $v^n(\Delta^*) = t$ by (1), (2), and (3). Therefore $v^n(\Delta^*\vee_t\sim_t\neg\alpha) = t$. Second, we show that $\Gamma_*\rightarrow\Delta^*\vee_t\sim_t\beta$ is a tautology. This case is similar to the proof above, since we can derive $v^t(\beta) = f$.

• (43): Suppose that $\sim_f(\alpha\rightarrow\beta)\wedge_t\Gamma_*\rightarrow\Delta^*$ is a tautology. First, we show that $\Gamma_* \rightarrow \Delta^*\vee_t\sim_f\alpha$ is a tautology. Suppose that (1) $v^n(\Gamma_*) = t$. We show $v^n(\Delta^*\vee_t\sim_f\alpha) = t$. If $v^n(\sim_f\alpha) = t$, then $v^n(\Delta^*\vee_t\sim_f\alpha) = t$. Thus, suppose that $v^n(\sim_f\alpha) = v^f(\alpha) = f$. Then, $v^f(\alpha\rightarrow\beta) = t$, and hence (2) $v^n(\sim_f(\alpha\rightarrow\beta)) = v^f(\alpha\rightarrow\beta) = t$. On the other hand, we have (3) $v^n(\sim_f(\alpha\rightarrow\beta)\wedge_t\Gamma_*\rightarrow\Delta^*) = t$ by the hypothesis. Thus, we obtain $v^n(\Delta^*) = t$ by (1), (2) and (3). Therefore $v^n(\Delta^*\vee_t\sim_f\alpha) = t$. Second, we show that $\sim_f\beta\wedge_t\Gamma_*\rightarrow\Delta^*$ is a tautology. Suppose that (4) $v^n(\sim_f\beta\wedge_t\Gamma_*) = t$. Then, (5) $v^n(\sim_f\beta) = v^f(\beta) = t$ and (6) $v^n(\Gamma_*) = t$. By (5), we have $v^f(\alpha\rightarrow\beta) = t$, and hence (7) $v^n(\sim_f(\alpha\rightarrow\beta)) = v^f(\alpha\rightarrow\beta) = t$. By (3), (6), and (7), we obtain the required fact $v^n(\Delta^*) = t$. □

Lemma 17. *Let $\sim_*$ be $\sim_t$, $\sim_f$, or $\sim_b$.*

(1) *Suppose that each α_i or β_j in $\{\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n\}$ is a propositional variable or a formula of the form $\sim_*\gamma$ where γ is a propositional variable. Then, the sequent $\alpha_1, \ldots, \alpha_m \Rightarrow \beta_1, \ldots, \beta_n$ is a tautology if and only if* (a) *there are α_i $(i \le m)$ and β_j $(j \le n)$ such that $\alpha_i \equiv \beta_j$, or* (b) *there are α_i $(i \le m)$ and β_j $(j \le n)$ such that ($\alpha_i \equiv {\sim_f}{\sim_t}p$ and $\beta_j \equiv {\sim_t}{\sim_f}p$) or ($\alpha_i \equiv {\sim_t}{\sim_f}p$ and $\beta_j \equiv {\sim_f}{\sim_t}p$) where p is a propositional variable.*

(2) *Sequents of the form $(\alpha, \Gamma \Rightarrow \Delta, \alpha)$, $({\sim_f}{\sim_t}\alpha, \Gamma \Rightarrow \Delta, {\sim_t}{\sim_f}\alpha)$, and $({\sim_t}{\sim_f}\alpha, \Gamma \Rightarrow \Delta, {\sim_f}{\sim_t}\alpha)$ are provable in cut-free* L_{16}.

Proof. We show only (1). Suppose there are no α_i and β_j such that (a) and (b). We specify a valuation v^n as follows: $v^n(\alpha_i) = t$ $(i = 1, \ldots, m)$ and $v^n(\beta_j) = f$ $(j = 1, \ldots, n)$. Then we obtain $v^n((\alpha_1 \wedge_t \cdots \wedge_t \alpha_m) \to (\beta_1 \vee_t \cdots \vee_t \beta_n)) = f$, and hence $\alpha_1, \ldots, \alpha_m \Rightarrow \beta_1, \ldots, \beta_n$ is not a tautology. Conversely, suppose there are α_i and β_j such that (a) and (b). Then $\alpha_1, \ldots, \alpha_m \Rightarrow \beta_1, \ldots, \beta_n$ is a tautology. $\Box$

Lemma 18. *Let S_1 (or S_1 ; S_2) be a decomposition of S. If S_1 (or S_2) is provable in cut-free* L_{16}, *then so is S.*

Theorem 19 (*Strong Completeness for* L_{16}). *For any sequent S, if S is a tautology, then* $L_{16} - (\text{cut}) \vdash S$.

Proof. Suppose that a sequent S is a tautology. We can obtain a complete decomposition tree of S by Lemma 15. All the leaves of this complete decomposition tree are tautologies by using Lemma 16 repeatedly. Then, these leaves are provable in cut-free L_{16} by Lemma 17(1) and (2). By using Lemma 18 repeatedly for the complete decomposition tree of S, all the sequents in the tree are provable in cut-free L_{16}. Therefore, in particular, S is provable in cut-free L_{16}. $\Box$

Theorem 20 (*Cut-Elimination for* L_{16}). *The rule* (cut) *is admissible in cut-free* L_{16}.

Proof. Suppose $L_{16} \vdash S$. Then, S is a tautology by Theorem 12. By Theorem 19, we obtain $L_{16} - (\text{cut}) \vdash S$. $\Box$

Odintsov [15] proved that L_B is the set of all formulas α of the language under consideration such that for every assignment v from the set of propositional variables into **16**, $v_b(\alpha) = 1$. (Also, $L_B = \{\alpha \mid \forall v(v_n(\alpha) = 0\}$.) Note that (*) for the two negations $\sim_t$ and $\sim_f$, the valuations v^n, v^t, v^f, and v^b are defined like Odintsov's v_n, v_t, v_f, and v_b, respectively. Moreover, (**) for the other connectives, the valuations v^n and v^f are defined like Odintsov's v_t and v_b (and v^t, v^b are defined like v_n and v_f). The following theorem shows how the valuation functions v^n, v^f, v^t, and v^b are related to Odintsov's co-ordinate valuations.

Theorem 21. *For every formula α,*

(1) $(\forall v^n, v^t, v^f, v^b : v^n(\alpha) = t)$ *iff* $(\forall v_n, v_t, v_f, v_b : v_b(\alpha) = 1)$,
(2) $(\forall v^n, v^t, v^f, v^b : v^f(\alpha) = t)$ *iff* $(\forall v_n, v_t, v_f, v_b : v_t(\alpha) = 1)$,
(3) $(\forall v^n, v^t, v^f, v^b : v^b(\alpha) = t)$ *iff* $(\forall v_n, v_t, v_f, v_b : v_n(\alpha) = 1)$,
(4) $(\forall v^n, v^t, v^f, v^b : v^t(\alpha) = t)$ *iff* $(\forall v_n, v_t, v_f, v_b : v_f(\alpha) = 1)$.

Proof. By simultaneous induction on α. For atoms and negated atoms the claims hold trivially. For negated complex formulas, the claims hold by (*) and the four induction hypotheses. We consider here two cases for Claim 1.
$v^n({\sim_f}(\alpha \wedge_t \beta)) = t$ iff $v^f(\alpha \wedge_t \beta) = t$ iff ($v^f(\alpha) = t$ and $v^f(\beta) = t$) iff $v_t(\alpha) = 1$ and $v_t(\beta) = 1$) (induction hypothesis for 2) iff $v_t(\alpha \wedge_t \beta) = 1$ iff $v_b({\sim_f}(\alpha \wedge_t \beta)) = 1$.

$v^n({\sim_t}{\sim_f}\alpha) = t$ iff $v^t({\sim_f}\alpha) = t$ iff $v^b(\alpha) = t$ iff (induction hypothesis for 3) $v_n(\alpha) = 1$ iff $v_f({\sim_f}\alpha) = 1$ iff $v_b({\sim_t}{\sim_f}\alpha) = 1$.

For formulas of the form $(\beta \sharp \delta)$, where $\sharp$ is a binary connective, the claims follow by (**) and the respective induction hypothesis. We consider here one case for Claim 2: $v^f(\alpha \to \beta) = t$ iff ($v^f(\alpha) = f$ or $v^f(\beta) = t$) iff ($v_t(\alpha) = 0$ or $v_t(\beta) = 1$) iff $v_t(\alpha \to \beta) = 1$. $\Box$

4. First-order case

4.1. Sequent calculus

We will extend the sequent system L_{16} to a first-order sequent calculus F_{16}. The notational conventions of the previous section are also adopted in this section. To begin with, we introduce the first-order language $\mathcal{L}$, in which the quantifiers come in two versions, one pair of quantifiers is related to truth, the other pair is related to falsity. For the sake of simplicity of the discussion, $\mathcal{L}$ is a language without individual constants and function symbols. *Formulas* are constructed from countably many predicate symbols $p, q, \ldots$, countably many individual variables $x, y, \ldots$, and the logical connectives $\to$, $\neg$, $\wedge_t$, $\vee_t$, $\wedge_f$, $\vee_f$, $\sim_t$, $\sim_f$, $\forall_t$, $\forall_f$, $\exists_t$, and $\exists_f$. An expression $\alpha[y/x]$ means the formula which is obtained from the formula α by replacing all free occurrences of the individual variable x in α by the individual variable y, but avoiding a clash of variables by a suitable renaming of bound variables. A 0-ary predicate is regarded as a propositional variable.

Definition 22 (F_{16}). Let $\sim_d \in \{\sim_t\sim_t, \sim_f\sim_f, \sim_b\sim_b\}$ and $\sim_e \in \{\sim_t, \sim_b\}$.
The sequent calculus F_{16} is obtained from L_{16} by adding the quantifier inference rules of the form:

$$\frac{\alpha[y/x], \Gamma \Rightarrow \Delta}{\forall_t x\alpha, \Gamma \Rightarrow \Delta}\ (\forall_t l) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha[z/x]}{\Gamma \Rightarrow \Delta, \forall_t x\alpha}\ (\forall_t r)$$
$$\frac{\alpha[z/x], \Gamma \Rightarrow \Delta}{\exists_t x\alpha, \Gamma \Rightarrow \Delta}\ (\exists_t l) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha[y/x]}{\Gamma \Rightarrow \Delta, \exists_t x\alpha}\ (\exists_t r)$$
$$\frac{\alpha[z/x], \Gamma \Rightarrow \Delta}{\forall_f x\alpha, \Gamma \Rightarrow \Delta}\ (\forall_f l) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha[y/x]}{\Gamma \Rightarrow \Delta, \forall_f x\alpha}\ (\forall_f r)$$
$$\frac{\alpha[y/x], \Gamma \Rightarrow \Delta}{\exists_f x\alpha, \Gamma \Rightarrow \Delta}\ (\exists_f l) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha[z/x]}{\Gamma \Rightarrow \Delta, \exists_f x\alpha}\ (\exists_f r)$$
$$\frac{\sim_e\alpha[z/x], \Gamma \Rightarrow \Delta}{\sim_e\forall_t x\alpha, \Gamma \Rightarrow \Delta}\ (\sim_e\forall_t l) \qquad \frac{\Gamma \Rightarrow \Delta, \sim_e\alpha[y/x]}{\Gamma \Rightarrow \Delta, \sim_e\forall_t x\alpha}\ (\sim_e\forall_t r)$$
$$\frac{\sim_e\alpha[y/x], \Gamma \Rightarrow \Delta}{\sim_e\exists_t x\alpha, \Gamma \Rightarrow \Delta}\ (\sim_e\exists_t l) \qquad \frac{\Gamma \Rightarrow \Delta, \sim_e\alpha[z/x]}{\Gamma \Rightarrow \Delta, \sim_e\exists_t x\alpha}\ (\sim_e\exists_t r)$$
$$\frac{\sim_e\alpha[y/x], \Gamma \Rightarrow \Delta}{\sim_e\forall_f x\alpha, \Gamma \Rightarrow \Delta}\ (\sim_e\forall_f l) \qquad \frac{\Gamma \Rightarrow \Delta, \sim_e\alpha[z/x]}{\Gamma \Rightarrow \Delta, \sim_e\forall_f x\alpha}\ (\sim_e\forall_f r)$$
$$\frac{\sim_e\alpha[z/x], \Gamma \Rightarrow \Delta}{\sim_e\exists_f x\alpha, \Gamma \Rightarrow \Delta}\ (\sim_e\exists_f l) \qquad \frac{\Gamma \Rightarrow \Delta, \sim_e\alpha[y/x]}{\Gamma \Rightarrow \Delta, \sim_e\exists_f x\alpha}\ (\sim_e\exists_f r)$$
$$\frac{\sim_f\alpha[y/x], \Gamma \Rightarrow \Delta}{\sim_f\forall_t x\alpha, \Gamma \Rightarrow \Delta}\ (\sim_f\forall_t l) \qquad \frac{\Gamma \Rightarrow \Delta, \sim_f\alpha[z/x]}{\Gamma \Rightarrow \Delta, \sim_f\forall_t x\alpha}\ (\sim_f\forall_t r)$$
$$\frac{\sim_f\alpha[z/x], \Gamma \Rightarrow \Delta}{\sim_f\exists_t x\alpha, \Gamma \Rightarrow \Delta}\ (\sim_f\exists_t l) \qquad \frac{\Gamma \Rightarrow \Delta, \sim_f\alpha[y/x]}{\Gamma \Rightarrow \Delta, \sim_f\exists_t x\alpha}\ (\sim_f\exists_t r)$$
$$\frac{\sim_f\alpha[z/x], \Gamma \Rightarrow \Delta}{\sim_f\forall_f x\alpha, \Gamma \Rightarrow \Delta}\ (\sim_f\forall_f l) \qquad \frac{\Gamma \Rightarrow \Delta, \sim_f\alpha[y/x]}{\Gamma \Rightarrow \Delta, \sim_f\forall_f x\alpha}\ (\sim_f\forall_f r)$$
$$\frac{\sim_f\alpha[y/x], \Gamma \Rightarrow \Delta}{\sim_f\exists_f x\alpha, \Gamma \Rightarrow \Delta}\ (\sim_f\exists_f l) \qquad \frac{\Gamma \Rightarrow \Delta, \sim_f\alpha[z/x]}{\Gamma \Rightarrow \Delta, \sim_f\exists_f x\alpha}\ (\sim_f\exists_f r)$$

where y is an arbitrary individual variable, and z is an individual variable which has the eigenvariable condition, i.e., z does not occur as a free individual variable in the lower sequent of the rule.

Note that, for example, $L_{16} - (\text{cut}) \vdash \exists_f x\, p(x) \Rightarrow \forall_f x\, p(x)$, whereas the eigenvariable condition prevents $\exists_t x\, p(x) \Rightarrow \forall_t x\, p(x)$ from being provable.
Propositions 6 and 7 hold for F_{16}.

4.2. Semantics

Definition 23. A structure $\mathcal{A} := \langle U, I^n, I^t, I^f, I^b\rangle$ is called a *model* if the following conditions hold:

(1) U is a non-empty set,
(2) I^n, I^t, I^f and I^b are mappings such that $p^{I^n}, p^{I^t}, p^{I^f}, p^{I^b} \subseteq U^n$ (i.e. $p^{I^n}, p^{I^t}, p^{I^f}$ and p^{I^b} are n-ary relations on U) for an n-ary predicate symbol p.

We introduce the notation $\underline{u}$ as the name of $u \in U$, and we denote as $\mathcal{L}[\mathcal{A}]$ the language obtained from $\mathcal{L}$ by adding the names of all the elements of U. A formula α is called a *closed formula* if α has no free individual variable. A formula of the form $\forall_t x_1 \cdots \forall_t x_m \alpha$ is called the *universal closure* of α if the free variables of α are $x_1, \ldots, x_m$. We write $cl(\alpha)$ for the universal closure of α.

Definition 24. Let $\mathcal{A} := \langle U, I^n, I^t, I^f, I^b\rangle$ be a model. The satisfaction relations $\mathcal{A} \models^n \alpha$, $\mathcal{A} \models^t \alpha$, $\mathcal{A} \models^f \alpha$, and $\mathcal{A} \models^b \alpha$ for any closed formula α of $\mathcal{L}[\mathcal{A}]$ are defined inductively as follows: for any $* \in \{n, t, f, b\}$, any $e \in \{t, b\}$, and any $g \in \{f, n\}$,

(1) $[\mathcal{A} \models^* p(\underline{x}_1, \ldots, \underline{x}_n)$ iff $(x_1, \ldots, x_n) \in p^{I^*}]$ for any n-ary atomic formula $p(\underline{x}_1, \ldots, \underline{x}_n)$,
(2) $\mathcal{A} \models^g \alpha \rightarrow \beta$ iff not-($\mathcal{A} \models^g \alpha$) or $\mathcal{A} \models^g \beta$,
(3) $\mathcal{A} \models^g \neg\alpha$ iff not-($\mathcal{A} \models^g \alpha$),
(4) $\mathcal{A} \models^g \alpha \wedge_t \beta$ iff $\mathcal{A} \models^g \alpha$ and $\mathcal{A} \models^g \beta$,
(5) $\mathcal{A} \models^g \alpha \vee_t \beta$ iff $\mathcal{A} \models^g \alpha$ or $\mathcal{A} \models^g \beta$,
(6) $\mathcal{A} \models^g \alpha \wedge_f \beta$ iff $\mathcal{A} \models^g \alpha$ or $\models^g \beta$,
(7) $\mathcal{A} \models^g \alpha \vee_f \beta$ iff $\mathcal{A} \models^g \alpha$ and $\mathcal{A} \models^g \beta$,
(8) $\mathcal{A} \models^g \forall_t x\alpha$ iff $\mathcal{A} \models^g \alpha[\underline{u}/x]$ for all $u \in U$,
(9) $\mathcal{A} \models^g \exists_t x\alpha$ iff $\mathcal{A} \models^g \alpha[\underline{u}/x]$ for some $u \in U$,
(10) $\mathcal{A} \models^g \forall_f x\alpha$ iff $\mathcal{A} \models^g \alpha[\underline{u}/x]$ for some $u \in U$,
(11) $\mathcal{A} \models^g \exists_f x\alpha$ iff $\mathcal{A} \models^g \alpha[\underline{u}/x]$ for all $u \in U$,

(12) $\mathcal{A} \models^n \sim_t\alpha$ iff $\mathcal{A} \models^t \alpha$,
(13) $\mathcal{A} \models^n \sim_f\alpha$ iff $\mathcal{A} \models^f \alpha$,
(14) $\mathcal{A} \models^n \sim_b\alpha$ iff $\mathcal{A} \models^b \alpha$,
(15) $\mathcal{A} \models^e \alpha\to\beta$ iff not-($\mathcal{A} \models^e \alpha$) and $\mathcal{A} \models^e \beta$,
(16) $\mathcal{A} \models^e \neg\alpha$ iff not-($\mathcal{A} \models^e \alpha$),
(17) $\mathcal{A} \models^e \alpha\wedge_t\beta$ iff $\mathcal{A} \models^e \alpha$ or $\mathcal{A} \models^e \beta$,
(18) $\mathcal{A} \models^e \alpha\vee_t\beta$ iff $\mathcal{A} \models^e \alpha$ and $\mathcal{A} \models^e \beta$,
(19) $\mathcal{A} \models^e \alpha\wedge_f\beta$ iff $\mathcal{A} \models^e \alpha$ and $\mathcal{A} \models^e \beta$,
(20) $\mathcal{A} \models^e \alpha\vee_f\beta$ iff $\mathcal{A} \models^e \alpha$ or $\mathcal{A} \models^e \beta$,
(21) $\mathcal{A} \models^e \forall_t x\alpha$ iff $\mathcal{A} \models^e \alpha[\underline{u}/x]$ for some $u \in U$,
(22) $\mathcal{A} \models^e \exists_t x\alpha$ iff $\mathcal{A} \models^e \alpha[\underline{u}/x]$ for all $u \in U$,
(23) $\mathcal{A} \models^e \forall_f x\alpha$ iff $\mathcal{A} \models^e \alpha[\underline{u}/x]$ for all $u \in U$,
(24) $\mathcal{A} \models^e \exists_f x\alpha$ iff $\mathcal{A} \models^e \alpha[\underline{u}/x]$ for some $u \in U$,
(25) $\mathcal{A} \models^t \sim_t\alpha$ iff $\mathcal{A} \models^n \alpha$,
(26) $\mathcal{A} \models^t \sim_f\alpha$ iff $\mathcal{A} \models^b \alpha$,
(27) $\mathcal{A} \models^t \sim_b\alpha$ iff $\mathcal{A} \models^f \alpha$,
(28) $\mathcal{A} \models^f \sim_t\alpha$ iff $\mathcal{A} \models^b \alpha$,
(29) $\mathcal{A} \models^f \sim_f\alpha$ iff $\mathcal{A} \models^n \alpha$,
(30) $\mathcal{A} \models^f \sim_b\alpha$ iff $\mathcal{A} \models^t \alpha$,
(31) $\mathcal{A} \models^b \sim_t\alpha$ iff $\mathcal{A} \models^f \alpha$,
(32) $\mathcal{A} \models^b \sim_f\alpha$ iff $\mathcal{A} \models^t \alpha$,
(33) $\mathcal{A} \models^b \sim_b\alpha$ iff $\mathcal{A} \models^n \alpha$.

The satisfaction relations $\mathcal{A} \models^* \alpha$ ($* \in \{n, t, f, b\}$) for any formula α of $\mathcal{L}$ are defined by ($\mathcal{A} \models^* \alpha$ iff $\mathcal{A} \models^* cl(\alpha)$). A formula α of $\mathcal{L}$ is called *valid* if $\mathcal{A} \models^n \alpha$ holds for any model $\mathcal{A}$. A sequent $\Gamma \Rightarrow \Delta$ of $\mathcal{L}$ is called valid if so is the formula $\Gamma_*\to\Delta^*$.

Note that the four interpretation functions I^n, I^f, I^t and I^b correspond to the valuation functions v^n, v^f, v^t and v^b, respectively. Moreover, it is noted that the following conditions hold: for any $* \in \{n, t, f, b\}$,

(1) $\mathcal{A} \models^* \alpha\wedge_t\beta$ iff $\mathcal{A} \models^* \alpha\vee_f\beta$,
(2) $\mathcal{A} \models^* \alpha\vee_t\beta$ iff $\mathcal{A} \models^* \alpha\wedge_f\beta$,
(3) $\mathcal{A} \models^* \forall_t\alpha$ iff $\mathcal{A} \models^* \exists_f\alpha$,
(4) $\mathcal{A} \models^* \exists_t\alpha$ iff $\mathcal{A} \models^* \forall_f\alpha$,
(5) $\mathcal{A} \models^t \sim_f\alpha$ iff $\mathcal{A} \models^f \sim_t\alpha$,
(6) $\mathcal{A} \models^t \alpha$ iff $\mathcal{A} \models^n \sim_t\alpha$ iff $\mathcal{A} \models^f \sim_b\alpha$ iff $\mathcal{A} \models^b \sim_f\alpha$,
(7) $\mathcal{A} \models^f \alpha$ iff $\mathcal{A} \models^n \sim_f\alpha$ iff $\mathcal{A} \models^t \sim_b\alpha$ iff $\mathcal{A} \models^b \sim_t\alpha$,
(8) $\mathcal{A} \models^b \alpha$ iff $\mathcal{A} \models^n \sim_b\alpha$ iff $\mathcal{A} \models^t \sim_f\alpha$ iff $\mathcal{A} \models^f \sim_t\alpha$.

Theorem 25 (*Soundness for* F_{16})**.** *For any sequent S, if* $\mathrm{F}_{16} \vdash S$*, then S is valid.*

Proof. By induction on the proof P of S. We distinguish the cases according to the last inference of P. We show only the following case.

Case ($\sim_t\exists_t$r): The last inference of P is of the form:

$$\frac{\Gamma \Rightarrow \Delta, \sim_t\alpha[z/x]}{\Gamma \Rightarrow \Delta, \sim_t\exists_t x\alpha}\ (\sim_t\exists_t\mathrm{r}).$$

We show that "$\Gamma \Rightarrow \Delta, \sim_t\alpha[z/x]$ is valid" implies "$\Gamma \Rightarrow \Delta, \sim_t\exists_t x\alpha$ is valid". By the hypothesis, (i): $\forall_t z_1 \cdots \forall_t z_n \forall_t z(\Gamma_*\to(\Delta^*\vee_t(\sim_t\alpha[z/x])))$ (where $z_1, \ldots, z_n$ are the free individual variables occurring in $\Gamma \Rightarrow \Delta, \sim_t\exists_t x\alpha$) is valid. We show that $\mathcal{A} \models^n \forall_t z_1 \cdots \forall_t z_n(\Gamma_*\to(\Delta^*\vee_t(\sim_t\exists_t x\alpha)))$ for any model $\mathcal{A} := \langle U, I^n, I^t, I^f, I^b\rangle$, i.e., we show that for any $u_1, \ldots, u_n \in U$, $\mathcal{A} \models^n \underline{\Gamma}_*\to(\underline{\Delta}^*\vee_t(\sim_t\exists_t x\underline{\alpha}))$, where $\underline{\Gamma}_*$, $\underline{\Delta}^*$ and $\underline{\alpha}$ are respectively obtained from Γ_*, Δ^* and α by replacing $z_1, \ldots, z_n$ by $\underline{u}_1, \ldots, \underline{u}_n$ Here, we note that $(\sim_t\exists_t x\alpha)[\underline{u}_1/z_1, \ldots, \underline{u}_n/z_n]$ (the result of the simultaneous substitution of z_i by $\underline{u}_i$ ($1 \le i \le n$)) is equivalent to $\sim_t\exists_t x(\alpha[\underline{u}_1/z_1, \ldots, \underline{u}_n/z_n])$, i.e., $\sim_t\exists_t x\underline{\alpha}$. By (i), we have $\mathcal{A} \models^n (\underline{\Gamma}_*\to(\underline{\Delta}^*\vee_t(\sim_t\underline{\alpha}[z/x])))[\underline{w}/z]$ for any $w \in U$. By the eigenvariable condition, z is not occurring freely in $\underline{\Gamma}_*$, $\underline{\Delta}^*$ and $\underline{\alpha}$. Thus, $\underline{\Gamma}_*[\underline{w}/z]$ and $\underline{\Delta}^*[\underline{w}/z]$ are equivalent to $\underline{\Gamma}_*$and $\underline{\Delta}^*$, respectively, and $\underline{\alpha}[z/x][\underline{w}/z]$ is equivalent to $\underline{\alpha}[\underline{w}/z][\underline{w}/x]$, i.e., $\underline{\alpha}[\underline{w}/x]$. Therefore, for any $w \in U$, we have that (a): $\mathcal{A} \models^n \underline{\Gamma}_*\to(\underline{\Delta}^*\vee_t\sim_t\underline{\alpha}[\underline{w}/x])$. Suppose that (b): [$\mathcal{A} \models^n \underline{\Gamma}_*$ and not ($\mathcal{A} \models^n \underline{\Delta}^*$)]. Then, by (a), we have that for any $w \in U$, $\mathcal{A} \models^n \sim_t\underline{\alpha}[\underline{w}/x]$, i.e., $\mathcal{A} \models^t \underline{\alpha}[\underline{w}/x]$. Therefore, we obtain (c): $\mathcal{A} \models^t \exists_t x\underline{\alpha}$, and hence $\mathcal{A} \models^n \sim_t\exists_t x\underline{\alpha}$. This means that (b) implies (c), i.e., $\mathcal{A} \models^n \underline{\Gamma}_*$ implies ($\mathcal{A} \models^n \underline{\Delta}^*$ or $\mathcal{A} \models^n \sim_t\exists_t x\underline{\alpha}$). Therefore, we have the required fact that $\mathcal{A} \models^n \underline{\Gamma}_*\to(\underline{\Delta}^*\vee_t(\sim_t\exists_t x\underline{\alpha}))$ for any $u_1, \ldots u_n \in U$. $\square$

4.3. Completeness and cut-elimination

In the following, we prove the (strong) completeness and cut-elimination theorems for F_{16} by using Schütte's method [23].

Definition 26. Let $\sim_d \in \{\sim_t\sim_t, \sim_f\sim_f, \sim_b\sim_b\}$ and $\sim_e \in \{\sim_t, \sim_b\}$. A sequent $\Gamma \Rightarrow \Delta$ is called *saturated* if for any formulas α and β,

(1) $\alpha\to\beta \in \Gamma$ implies ($\alpha \in \Delta$ or $\beta \in \Gamma$),
(2) $\alpha\to\beta \in \Delta$ implies ($\alpha \in \Gamma$ and $\beta \in \Delta$),
(3) $\neg\alpha \in \Gamma$ implies $\alpha \in \Delta$,
(4) $\neg\alpha \in \Delta$ implies $\alpha \in \Gamma$,
(5) $\alpha\wedge_t\beta \in \Gamma$ implies ($\alpha \in \Gamma$ and $\beta \in \Gamma$),
(6) $\alpha\wedge_t\beta \in \Delta$ implies ($\alpha \in \Delta$ or $\beta \in \Delta$),
(7) $\alpha\vee_t\beta \in \Gamma$ implies ($\alpha \in \Gamma$ or $\beta \in \Gamma$),
(8) $\alpha\vee_t\beta \in \Delta$ implies ($\alpha \in \Delta$ and $\beta \in \Delta$),
(9) $\alpha\wedge_f\beta \in \Gamma$ implies ($\alpha \in \Gamma$ or $\beta \in \Gamma$),
(10) $\alpha\wedge_f\beta \in \Delta$ implies ($\alpha \in \Delta$ and $\beta \in \Delta$),
(11) $\alpha\vee_f\beta \in \Gamma$ implies ($\alpha \in \Gamma$ and $\beta \in \Gamma$),
(12) $\alpha\vee_f\beta \in \Delta$ implies ($\alpha \in \Delta$ or $\beta \in \Delta$),
(13) $\forall_t x\alpha \in \Gamma$ implies ($\alpha[y/x] \in \Gamma$ for any individual variable y),
(14) $\forall_t x\alpha \in \Delta$ implies ($\alpha[z/x] \in \Delta$ for some individual variable z),
(15) $\exists_t x\alpha \in \Gamma$ implies ($\alpha[z/x] \in \Gamma$ for some individual variable z),
(16) $\exists_t x\alpha \in \Delta$ implies ($\alpha[y/x] \in \Delta$ for any individual variable y),
(17) $\forall_f x\alpha \in \Gamma$ implies ($\alpha[z/x] \in \Gamma$ for some individual variable z),
(18) $\forall_f x\alpha \in \Delta$ implies ($\alpha[y/x] \in \Delta$ for any individual variable y),
(19) $\exists_f x\alpha \in \Gamma$ implies ($\alpha[y/x] \in \Gamma$ for any individual variable y),
(20) $\exists_f x\alpha \in \Delta$ implies ($\alpha[z/x] \in \Delta$ for some individual variable z),
(21) $\sim_d\alpha \in \Gamma$ implies $\alpha \in \Gamma$,
(22) $\sim_d\alpha \in \Delta$ implies $\alpha \in \Delta$,
(23) $\sim_t\sim_f\sim_t\alpha \in \Gamma$ implies $\sim_f\alpha \in \Gamma$,
(24) $\sim_t\sim_f\sim_t\alpha \in \Delta$ implies $\sim_f\alpha \in \Delta$,
(25) $\sim_f\sim_t\sim_f\alpha \in \Gamma$ implies $\sim_t\alpha \in \Gamma$,
(26) $\sim_f\sim_t\sim_f\alpha \in \Delta$ implies $\sim_t\alpha \in \Delta$,
(27) $\sim_t\neg\sim_t\alpha \in \Gamma$ implies $\neg\alpha \in \Gamma$,
(28) $\sim_t\neg\sim_t\alpha \in \Delta$ implies $\neg\alpha \in \Delta$,
(29) $\sim_f\neg\sim_f\alpha \in \Gamma$ implies $\neg\alpha \in \Gamma$,
(30) $\sim_f\neg\sim_f\alpha \in \Delta$ implies $\neg\alpha \in \Delta$,
(31) $\sim_f\sim_t\sim_t\alpha \in \Gamma$ implies $\sim_f\alpha \in \Gamma$,
(32) $\sim_f\sim_t\sim_t\alpha \in \Delta$ implies $\sim_f\alpha \in \Delta$,
(33) $\sim_t\sim_f\sim_f\alpha \in \Gamma$ implies $\sim_t\alpha \in \Gamma$,
(34) $\sim_t\sim_f\sim_f\alpha \in \Delta$ implies $\sim_t\alpha \in \Delta$,
(35) $\neg\sim_t\sim_t\alpha \in \Gamma$ implies $\neg\alpha \in \Gamma$,
(36) $\neg\sim_t\sim_t\alpha \in \Delta$ implies $\neg\alpha \in \Delta$,
(37) $\neg\sim_f\sim_f\alpha \in \Gamma$ implies $\neg\alpha \in \Gamma$,
(38) $\neg\sim_f\sim_f\alpha \in \Delta$ implies $\neg\alpha \in \Delta$,
(39) $\sim_e(\alpha\to\beta) \in \Gamma$ implies ($\sim_e\neg\alpha \in \Gamma$ and $\sim_e\beta \in \Gamma$),
(40) $\sim_e(\alpha\to\beta) \in \Delta$ implies ($\sim_e\neg\alpha \in \Delta$ or $\sim_e\beta \in \Delta$),
(41) $\sim_e\neg\alpha \in \Gamma$ implies $\sim_e\alpha \in \Delta$,
(42) $\sim_e\neg\alpha \in \Delta$ implies $\sim_e\alpha \in \Gamma$,
(43) $\sim_e(\alpha\wedge_t\beta) \in \Gamma$ implies ($\sim_e\alpha \in \Gamma$ or $\sim_e\beta \in \Gamma$),
(44) $\sim_e(\alpha\wedge_t\beta) \in \Delta$ implies ($\sim_e\alpha \in \Delta$ and $\sim_e\beta \in \Delta$),
(45) $\sim_e(\alpha\vee_t\beta) \in \Gamma$ implies ($\sim_e\alpha \in \Gamma$ and $\sim_e\beta \in \Gamma$),
(46) $\sim_e(\alpha\vee_t\beta) \in \Delta$ implies ($\sim_e\alpha \in \Delta$ or $\sim_e\beta \in \Delta$),
(47) $\sim_e(\alpha\wedge_f\beta) \in \Gamma$ implies ($\sim_e\alpha \in \Gamma$ and $\sim_e\beta \in \Gamma$),
(48) $\sim_e(\alpha\wedge_f\beta) \in \Delta$ implies ($\sim_e\alpha \in \Delta$ or $\sim_e\beta \in \Delta$),
(49) $\sim_e(\alpha\vee_f\beta) \in \Gamma$ implies ($\sim_e\alpha \in \Gamma$ or $\sim_e\beta \in \Gamma$),
(50) $\sim_e(\alpha\vee_f\beta) \in \Delta$ implies ($\sim_e\alpha \in \Delta$ and $\sim_e\beta \in \Delta$),
(51) $\sim_e\forall_t x\alpha \in \Gamma$ implies ($\sim_e\alpha[z/x] \in \Gamma$ for some individual variable z),
(52) $\sim_e\forall_t x\alpha \in \Delta$ implies ($\sim_e\alpha[y/x] \in \Delta$ for any individual variable y),
(53) $\sim_e\exists_t x\alpha \in \Gamma$ implies ($\sim_e\alpha[y/x] \in \Gamma$ for any individual variable y),
(54) $\sim_e\exists_t x\alpha \in \Delta$ implies ($\sim_e\alpha[z/x] \in \Delta$ for some individual variable z),

(55) $\sim_e\forall_f x\alpha \in \Gamma$ implies ($\sim_e\alpha[y/x] \in \Gamma$ for any individual variable y),
(56) $\sim_e\forall_f x\alpha \in \Delta$ implies ($\sim_e\alpha[z/x] \in \Delta$ for some individual variable z),
(57) $\sim_e\exists_f x\alpha \in \Gamma$ implies ($\sim_e\alpha[z/x] \in \Gamma$ for some individual variable z),
(58) $\sim_e\exists_f x\alpha \in \Delta$ implies ($\sim_e\alpha[y/x] \in \Delta$ for any individual variable y),
(59) $\sim_f(\alpha\to\beta) \in \Gamma$ implies ($\sim_f\alpha \in \Delta$ or $\sim_f\beta \in \Gamma$),
(60) $\sim_f(\alpha\to\beta) \in \Delta$ implies ($\sim_f\alpha \in \Gamma$ and $\sim_f\beta \in \Delta$),
(61) $\sim_f\neg\alpha \in \Gamma$ implies $\sim_f\alpha \in \Delta$,
(62) $\sim_f\neg\alpha \in \Delta$ implies $\sim_f\alpha \in \Gamma$,
(63) $\sim_f(\alpha\wedge_t\beta) \in \Gamma$ implies ($\sim_f\alpha \in \Gamma$ and $\sim_f\beta \in \Gamma$),
(64) $\sim_f(\alpha\wedge_t\beta) \in \Delta$ implies ($\sim_f\alpha \in \Delta$ or $\sim_f\beta \in \Delta$),
(65) $\sim_f(\alpha\vee_t\beta) \in \Gamma$ implies ($\sim_f\alpha \in \Gamma$ or $\sim_f\beta \in \Gamma$),
(66) $\sim_f(\alpha\vee_t\beta) \in \Delta$ implies ($\sim_f\alpha \in \Delta$ and $\sim_f\beta \in \Delta$),
(67) $\sim_f(\alpha\wedge_f\beta) \in \Gamma$ implies ($\sim_f\alpha \in \Gamma$ or $\sim_f\beta \in \Gamma$),
(68) $\sim_f(\alpha\wedge_f\beta) \in \Delta$ implies ($\sim_f\alpha \in \Delta$ and $\sim_f\beta \in \Delta$),
(69) $\sim_f(\alpha\vee_f\beta) \in \Gamma$ implies ($\sim_f\alpha \in \Gamma$ and $\sim_f\beta \in \Gamma$),
(70) $\sim_f(\alpha\vee_f\beta) \in \Delta$ implies ($\sim_f\alpha \in \Delta$ or $\sim_f\beta \in \Delta$),
(71) $\sim_f\forall_t x\alpha \in \Gamma$ implies ($\sim_f\alpha[y/x] \in \Gamma$ for any individual variable y),
(72) $\sim_f\forall_t x\alpha \in \Delta$ implies ($\sim_f\alpha[z/x] \in \Delta$ for some individual variable z),
(73) $\sim_f\exists_t x\alpha \in \Gamma$ implies ($\sim_f\alpha[z/x] \in \Gamma$ for some individual variable z),
(74) $\sim_f\exists_t x\alpha \in \Delta$ implies ($\sim_f\alpha[y/x] \in \Delta$ for any individual variable y),
(75) $\sim_f\forall_f x\alpha \in \Gamma$ implies ($\sim_f\alpha[z/x] \in \Gamma$ for some individual variable z),
(76) $\sim_f\forall_f x\alpha \in \Delta$ implies ($\sim_f\alpha[y/x] \in \Delta$ for any individual variable y),
(77) $\sim_f\exists_f x\alpha \in \Gamma$ implies ($\sim_f\alpha[y/x] \in \Gamma$ for any individual variable y),
(78) $\sim_f\exists_f x\alpha \in \Delta$ implies ($\sim_f\alpha[z/x] \in \Delta$ for some individual variable z).

We now generalize the notion of a sequent.

Definition 27. An expression $\Gamma \Rightarrow \Delta$ is called an *infinite sequent* if Γ and Δ are infinite (countable) sets of formulas. An infinite sequent $\Gamma \Rightarrow \Delta$ is called provable if a finite part $\Gamma' \Rightarrow \Delta'$ of the sequent is provable, i.e., Γ' and Δ' are finite subsets of Γ and Δ, respectively.

Definition 28. Let $\sim_d \in \{\sim_t\sim_t, \sim_f\sim_f, \sim_b\sim_b\}$ and $\sim_e \in \{\sim_t, \sim_b\}$. A *decomposition* of a sequent (or infinite sequent) S is defined as having the form S' or S'; S'' by

(1) $\alpha, \Gamma \Rightarrow \Delta, \alpha\to\beta, \beta$ is a decomposition of $\Gamma \Rightarrow \Delta, \alpha\to\beta$,
(2) $\alpha\to\beta, \Gamma \Rightarrow \Delta, \alpha$; $\beta, \alpha\to\beta, \Gamma \Rightarrow \Delta$ is a decomposition of $\alpha\to\beta, \Gamma \Rightarrow \Delta$,
(3) $\alpha, \Gamma \Rightarrow \Delta, \neg\alpha$ is a decomposition of $\Gamma \Rightarrow \Delta, \neg\alpha$,
(4) $\neg\alpha, \Gamma \Rightarrow \Delta, \alpha$ is a decomposition of $\neg\alpha, \Gamma \Rightarrow \Delta$,
(5) $\Gamma \Rightarrow \Delta, \alpha\wedge_t\beta, \alpha$; $\Gamma \Rightarrow \Delta, \alpha\wedge_t\beta, \beta$ is a decomposition of $\Gamma \Rightarrow \Delta, \alpha\wedge_t\beta$,
(6) $\alpha, \beta, \alpha\wedge_t\beta, \Gamma \Rightarrow \Delta$ is a decomposition of $\alpha\wedge_t\beta, \Gamma \Rightarrow \Delta$,
(7) $\Gamma \Rightarrow \Delta, \alpha\vee_t\beta, \alpha, \beta$ is a decomposition of $\Gamma \Rightarrow \Delta, \alpha\vee_t\beta$,
(8) $\alpha, \alpha\vee_t\beta, \Gamma \Rightarrow \Delta$; $\beta, \alpha\vee_t\beta, \Gamma \Rightarrow \Delta$ is a decomposition of $\alpha\vee_t\beta, \Gamma \Rightarrow \Delta$,
(9) $\Gamma \Rightarrow \Delta, \alpha\wedge_f\beta, \alpha, \beta$ is a decomposition of $\Gamma \Rightarrow \Delta, \alpha\wedge_f\beta$,
(10) $\alpha, \alpha\wedge_f\beta, \Gamma \Rightarrow \Delta$; $\beta, \alpha\wedge_f\beta, \Gamma \Rightarrow \Delta$ is a decomposition of $\alpha\wedge_f\beta, \Gamma \Rightarrow \Delta$,
(11) $\Gamma \Rightarrow \Delta, \alpha\vee_f\beta, \alpha$; $\Gamma \Rightarrow \Delta, \alpha\vee_f\beta, \beta$ is a decomposition of $\Gamma \Rightarrow \Delta, \alpha\vee_f\beta$,
(12) $\alpha, \beta, \alpha\vee_f\beta, \Gamma \Rightarrow \Delta$ is a decomposition of $\alpha\vee_f\beta, \Gamma \Rightarrow \Delta$,
(13) $\Gamma \Rightarrow \Delta, \forall_t x\alpha, \alpha[z/x]$ is a decomposition of $\Gamma \Rightarrow \Delta, \forall_t x\alpha$ where z is a *fresh* free individual variable, i.e., z is not occurring in $\Gamma \Rightarrow \Delta, \forall_t x\alpha$,
(14) $\alpha[y_1/x], \ldots, \alpha[y_m/x], \forall_t x\alpha, \Gamma \Rightarrow \Delta$ is a decomposition of $\forall_t x\alpha, \Gamma \Rightarrow \Delta$ where $y_1, \ldots, y_m$ are the free individual variables occurring in $\forall_t x\alpha, \Gamma \Rightarrow \Delta$, [2]
(15) $\Gamma \Rightarrow \Delta, \exists_t x\alpha, \alpha[y_1/x], \ldots, \alpha[y_m/x]$ is a decomposition of $\Gamma \Rightarrow \Delta, \exists_t x\alpha$ where $y_1, \ldots, y_m$ are the free individual variables occurring in $\Gamma \Rightarrow \Delta, \exists_t x\alpha$,
(16) $\alpha[z/x], \exists_t x\alpha, \Gamma \Rightarrow \Delta$ is a decomposition of $\exists_t x\alpha, \Gamma \Rightarrow \Delta$ where z is a fresh free individual variable,
(17) $\Gamma \Rightarrow \Delta, \forall_f x\alpha, \alpha[y_1/x], \ldots, \alpha[y_m/x]$ is a decomposition of $\Gamma \Rightarrow \Delta, \forall_f x\alpha$ where $y_1, \ldots, y_m$ are the free individual variables occurring in $\Gamma \Rightarrow \Delta, \forall_f x\alpha$,
(18) $\alpha[z/x], \forall_f x\alpha, \Gamma \Rightarrow \Delta$ is a decomposition of $\forall_f x\alpha, \Gamma \Rightarrow \Delta$ where z is a fresh free individual variable,
(19) $\Gamma \Rightarrow \Delta, \exists_f x\alpha, \alpha[z/x]$ is a decomposition of $\Gamma \Rightarrow \Delta, \exists_f x\alpha$ where z is a *fresh* free individual variable, i.e., z is not occurring in $\Gamma \Rightarrow \Delta, \exists_f x\alpha$,

[2] If $\forall_t x\alpha, \Gamma \Rightarrow \Delta$ has no free individual variable, then we adopt any free variable in $\mathcal{L}$. Such a condition is also adopted in the following corresponding items.

(20) $\alpha[y_1/x], \ldots, \alpha[y_m/x], \exists_f x\alpha, \Gamma \Rightarrow \Delta$ is a decomposition of $\exists_f x\alpha, \Gamma \Rightarrow \Delta$ where $y_1, \ldots, y_m$ are the free individual variables occurring in $\exists_f x\alpha, \Gamma \Rightarrow \Delta$,
(21) $\alpha, \sim_d\alpha, \Gamma \Rightarrow \Delta$ is a decomposition of $\sim_d\alpha, \Gamma \Rightarrow \Delta$,
(22) $\Gamma \Rightarrow \Delta, \sim_d\alpha, \alpha$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim_d\alpha$,
(23) $\sim_f\alpha, \sim_t\sim_f\sim_t\alpha, \Gamma \Rightarrow \Delta$ is a decomposition of $\sim_t\sim_f\sim_t\alpha, \Gamma \Rightarrow \Delta$,
(24) $\Gamma \Rightarrow \Delta, \sim_t\sim_f\sim_t\alpha, \sim_f\alpha$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim_t\sim_f\sim_t\alpha$,
(25) $\sim_t\alpha, \sim_f\sim_t\sim_f\alpha, \Gamma \Rightarrow \Delta$ is a decomposition of $\sim_f\sim_t\sim_f\alpha, \Gamma \Rightarrow \Delta$,
(26) $\Gamma \Rightarrow \Delta, \sim_f\sim_t\sim_f\alpha, \sim_t\alpha$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim_f\sim_t\sim_f\alpha$,
(27) $\neg\alpha, \sim_t\neg\sim_t\alpha, \Gamma \Rightarrow \Delta$ is a decomposition of $\sim_t\neg\sim_t\alpha, \Gamma \Rightarrow \Delta$,
(28) $\Gamma \Rightarrow \Delta, \sim_t\neg\sim_t\alpha, \neg\alpha$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim_t\neg\sim_t\alpha$,
(29) $\neg\alpha, \sim_f\neg\sim_f\alpha, \Gamma \Rightarrow \Delta$ is a decomposition of $\sim_f\neg\sim_f\alpha, \Gamma \Rightarrow \Delta$,
(30) $\Gamma \Rightarrow \Delta, \sim_f\neg\sim_f\alpha, \neg\alpha$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim_f\neg\sim_f\alpha$,
(31) $\sim_f\alpha, \sim_f\sim_t\sim_t\alpha, \Gamma \Rightarrow \Delta$ is a decomposition of $\sim_f\sim_t\sim_t\alpha, \Gamma \Rightarrow \Delta$,
(32) $\Gamma \Rightarrow \Delta, \sim_f\sim_t\sim_t\alpha, \sim_f\alpha$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim_f\sim_t\sim_t\alpha$,
(33) $\sim_t\alpha, \sim_t\sim_f\sim_f\alpha, \Gamma \Rightarrow \Delta$ is a decomposition of $\sim_t\sim_f\sim_f\alpha, \Gamma \Rightarrow \Delta$,
(34) $\Gamma \Rightarrow \Delta, \sim_t\sim_f\sim_f\alpha, \sim_t\alpha$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim_t\sim_f\sim_f\alpha$,
(35) $\neg\alpha, \neg\sim_t\sim_t\alpha, \Gamma \Rightarrow \Delta$ is a decomposition of $\neg\sim_t\sim_t\alpha, \Gamma \Rightarrow \Delta$,
(36) $\Gamma \Rightarrow \Delta, \neg\sim_t\sim_t\alpha, \sim_f\alpha$ is a decomposition of $\Gamma \Rightarrow \Delta, \neg\sim_t\sim_t\alpha$,
(37) $\neg\alpha, \neg\sim_f\sim_f\alpha, \Gamma \Rightarrow \Delta$ is a decomposition of $\neg\sim_f\sim_f\alpha, \Gamma \Rightarrow \Delta$,
(38) $\Gamma \Rightarrow \Delta, \neg\sim_f\sim_f\alpha, \neg\alpha$ is a decomposition of $\Gamma \Rightarrow \Delta, \neg\sim_f\sim_f\alpha$,
(39) $\Gamma \Rightarrow \Delta, \sim_e(\alpha\rightarrow\beta), \sim_e\neg\alpha\ ;\ \Gamma \Rightarrow \Delta, \sim_e(\alpha\rightarrow\beta), \sim_e\beta$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim_e(\alpha\rightarrow\beta)$,
(40) $\sim_e\neg\alpha, \sim_e\beta, \sim_e(\alpha\rightarrow\beta), \Gamma \Rightarrow \Delta$ is a decomposition of $\sim_e(\alpha\rightarrow\beta), \Gamma \Rightarrow \Delta$,
(41) $\sim_e\alpha, \Gamma \Rightarrow \Delta, \sim_e\neg\alpha$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim_e\neg\alpha$,
(42) $\sim_e\neg\alpha, \Gamma \Rightarrow \Delta, \sim_e\alpha$ is a decomposition of $\sim_e\neg\alpha, \Gamma \Rightarrow \Delta$,
(43) $\Gamma \Rightarrow \Delta, \sim_e(\alpha\wedge_t\beta), \sim_e\alpha, \sim_e\beta$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim_e(\alpha\wedge_t\beta)$,
(44) $\sim_e\alpha, \sim_e(\alpha\wedge_t\beta), \Gamma \Rightarrow \Delta\ ;\ \sim_e\beta, \sim_e(\alpha\wedge_t\beta), \Gamma \Rightarrow \Delta$ is a decomposition of $\sim_e(\alpha\wedge_t\beta), \Gamma \Rightarrow \Delta$,
(45) $\Gamma \Rightarrow \Delta, \sim_e(\alpha\vee_t\beta), \sim_e\alpha\ ;\ \Gamma \Rightarrow \Delta, \sim_e(\alpha\vee_t\beta), \sim_e\beta$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim_e(\alpha\vee_t\beta)$,
(46) $\sim_e\alpha, \sim_e\beta, \sim_e(\alpha\vee_t\beta), \Gamma \Rightarrow \Delta$ is a decomposition of $\sim_e(\alpha\vee_t\beta), \Gamma \Rightarrow \Delta$,
(47) $\Gamma \Rightarrow \Delta, \sim_e(\alpha\wedge_f\beta), \sim_e\alpha\ ;\ \Gamma \Rightarrow \Delta, \sim_e(\alpha\wedge_f\beta), \sim_e\beta$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim_e(\alpha\wedge_f\beta)$,
(48) $\sim_e\alpha, \sim_e\beta, \sim_e(\alpha\wedge_f\beta), \Gamma \Rightarrow \Delta$ is a decomposition of $\sim_e(\alpha\wedge_f\beta), \Gamma \Rightarrow \Delta$,
(49) $\Gamma \Rightarrow \Delta, \sim_e(\alpha\vee_f\beta), \sim_e\alpha, \sim_e\beta$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim_e(\alpha\vee_f\beta)$,
(50) $\sim_e\alpha, \sim_e(\alpha\vee_f\beta), \Gamma \Rightarrow \Delta\ ;\ \sim_e\beta, \sim_e(\alpha\vee_f\beta), \Gamma \Rightarrow \Delta$ is a decomposition of $\sim_e(\alpha\vee_f\beta), \Gamma \Rightarrow \Delta$,
(51) $\Gamma \Rightarrow \Delta, \sim_e\forall_t x\alpha, \sim_e\alpha[y_1/x], \ldots, \sim_e\alpha[y_m/x]$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim_e\forall_t x\alpha$ where $y_1, \ldots, y_m$ are the free individual variables occurring in $\Gamma \Rightarrow \Delta, \sim_e\forall_t x\alpha$,
(52) $\sim_e\alpha[z/x], \sim_e\forall_t x\alpha, \Gamma \Rightarrow \Delta$ is a decomposition of $\sim_e\forall_t x\alpha, \Gamma \Rightarrow \Delta$ where z is a fresh free individual variable,
(53) $\Gamma \Rightarrow \Delta, \sim_e\exists_t x\alpha, \sim_e\alpha[z/x]$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim_e\exists_t x\alpha$ where z is a fresh free individual variable,
(54) $\sim_e\alpha[y_1/x], \ldots, \sim_e\alpha[y_m/x], \sim_e\exists_t x\alpha, \Gamma \Rightarrow \Delta$ is a decomposition of $\sim_e\exists_t x\alpha, \Gamma \Rightarrow \Delta$ where $y_1, \ldots, y_m$ are the free individual variables occurring in $\sim_e\exists_t x\alpha, \Gamma \Rightarrow \Delta$,
(55) $\Gamma \Rightarrow \Delta, \sim_e\forall_f x\alpha, \sim_e\alpha[z/x]$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim_e\forall_f x\alpha$ where z is a fresh free individual variable,
(56) $\sim_e\alpha[y_1/x], \ldots, \sim_e\alpha[y_m/x], \sim_e\forall_f x\alpha, \Gamma \Rightarrow \Delta$ is a decomposition of $\sim_e\forall_f x\alpha, \Gamma \Rightarrow \Delta$ where $y_1, \ldots, y_m$ are the free individual variables occurring in $\sim_e\forall_f x\alpha, \Gamma \Rightarrow \Delta$,
(57) $\Gamma \Rightarrow \Delta, \sim_e\exists_f x\alpha, \sim_e\alpha[y_1/x], \ldots, \sim_e\alpha[y_m/x]$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim_e\exists_f x\alpha$ where $y_1, \ldots, y_m$ are the free individual variables occurring in $\Gamma \Rightarrow \Delta, \sim_e\exists_f x\alpha$,
(58) $\sim_e\alpha[z/x], \sim_e\exists_f x\alpha, \Gamma \Rightarrow \Delta$ is a decomposition of $\sim_e\exists_f x\alpha, \Gamma \Rightarrow \Delta$ where z is a fresh free individual variable,
(59) $\sim_f\alpha, \Gamma \Rightarrow \Delta, \sim_f(\alpha\rightarrow\beta), \sim_f\beta$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim_f(\alpha\rightarrow\beta)$,
(60) $\sim_f(\alpha\rightarrow\beta), \Gamma \Rightarrow \Delta, \sim_f\alpha\ ;\ \sim_f\beta, \sim_f(\alpha\rightarrow\beta), \Gamma \Rightarrow \Delta$ is a decomposition of $\sim_f(\alpha\rightarrow\beta), \Gamma \Rightarrow \Delta$,
(61) $\sim_f\alpha, \Gamma \Rightarrow \Delta, \sim_f\neg\alpha$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim_f\neg\alpha$,
(62) $\sim_f\neg\alpha, \Gamma \Rightarrow \Delta, \sim_f\alpha$ is a decomposition of $\sim_f\neg\alpha, \Gamma \Rightarrow \Delta$,
(63) $\Gamma \Rightarrow \Delta, \sim_f(\alpha\wedge_t\beta), \sim_f\alpha\ ;\ \Gamma \Rightarrow \Delta, \sim_f(\alpha\wedge_t\beta), \sim_f\beta$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim_f(\alpha\wedge_t\beta)$,
(64) $\sim_f\alpha, \sim_f\beta, \sim_f(\alpha\wedge_t\beta), \Gamma \Rightarrow \Delta$ is a decomposition of $\sim_f(\alpha\wedge_t\beta), \Gamma \Rightarrow \Delta$,
(65) $\Gamma \Rightarrow \Delta, \sim_f(\alpha\vee_t\beta), \sim_f\alpha, \sim_f\beta$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim_f(\alpha\vee_t\beta)$,
(66) $\sim_f\alpha, \sim_f(\alpha\vee_t\beta), \Gamma \Rightarrow \Delta\ ;\ \sim_f\beta, \sim_f(\alpha\vee_t\beta), \Gamma \Rightarrow \Delta$ is a decomposition of $\sim_f(\alpha\vee_t\beta), \Gamma \Rightarrow \Delta$,
(67) $\Gamma \Rightarrow \Delta, \sim_f(\alpha\wedge_f\beta), \sim_f\alpha, \sim_f\beta$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim_f(\alpha\wedge_f\beta)$,
(68) $\sim_f\alpha, \sim_f(\alpha\wedge_f\beta), \Gamma \Rightarrow \Delta\ ;\ \sim_f\beta, \sim_f(\alpha\wedge_f\beta), \Gamma \Rightarrow \Delta$ is a decomposition of $\sim_f(\alpha\wedge_f\beta), \Gamma \Rightarrow \Delta$,
(69) $\Gamma \Rightarrow \Delta, \sim_f(\alpha\vee_f\beta), \sim_f\alpha\ ;\ \Gamma \Rightarrow \Delta, \sim_f(\alpha\vee_f\beta), \sim_f\beta$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim_f(\alpha\vee_f\beta)$,
(70) $\sim_f\alpha, \sim_f\beta, \sim_f(\alpha\vee_f\beta), \Gamma \Rightarrow \Delta$ is a decomposition of $\sim_f(\alpha\vee_f\beta), \Gamma \Rightarrow \Delta$,
(71) $\Gamma \Rightarrow \Delta, \sim_f\forall_t x\alpha, \sim_f\alpha[z/x]$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim_f\forall_t x\alpha$ where z is a *fresh* free individual variable, i.e., z is not occurring in $\Gamma \Rightarrow \Delta, \sim_f\forall_t x\alpha$,
(72) $\sim_f\alpha[y_1/x], \ldots, \sim_f\alpha[y_m/x], \sim_f\forall_t x\alpha, \Gamma \Rightarrow \Delta$ is a decomposition of $\sim_f\forall_t x\alpha, \Gamma \Rightarrow \Delta$ where $y_1, \ldots, y_m$ are the free individual variables occurring in $\sim_f\forall_t x\alpha, \Gamma \Rightarrow \Delta$,

(73) $\Gamma \Rightarrow \Delta, \sim_f \exists_t x\alpha, \sim_f \alpha[y_1/x], \ldots, \sim_f \alpha[y_m/x]$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim_f \exists_t x\alpha$ where $y_1, \ldots, y_m$ are the free individual variables occurring in $\Gamma \Rightarrow \Delta, \sim_f \exists_t x\alpha$,
(74) $\sim_f \alpha[z/x], \sim_f \exists_t x\alpha, \Gamma \Rightarrow \Delta$ is a decomposition of $\sim_f \exists_t x\alpha, \Gamma \Rightarrow \Delta$ where z is a fresh free individual variable,
(75) $\Gamma \Rightarrow \Delta, \sim_f \forall_f x\alpha, \sim_f \alpha[y_1/x], \ldots, \sim_f \alpha[y_m/x]$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim_f \forall_f x\alpha$ where $y_1, \ldots, y_m$ are the free individual variables occurring in $\Gamma \Rightarrow \Delta, \sim_f \forall_f x\alpha$,
(76) $\sim_f \alpha[z/x], \sim_f \forall_f x\alpha, \Gamma \Rightarrow \Delta$ is a decomposition of $\sim_f \forall_f x\alpha, \Gamma \Rightarrow \Delta$ where z is a fresh free individual variable,
(77) $\Gamma \Rightarrow \Delta, \sim_f \exists_f x\alpha, \sim_f \alpha[z/x]$ is a decomposition of $\Gamma \Rightarrow \Delta, \sim_f \exists_f x\alpha$ where z is a *fresh* free individual variable, i.e., z is not occurring in $\Gamma \Rightarrow \Delta, \sim_f \exists_f x\alpha$,
(78) $\sim_f \alpha[y_1/x], \ldots, \sim_f \alpha[y_m/x], \sim_f \exists_f x\alpha, \Gamma \Rightarrow \Delta$ is a decomposition of $\sim_f \exists_f x\alpha, \Gamma \Rightarrow \Delta$ where $y_1, \ldots, y_m$ are the free individual variables occurring in $\sim_f \exists_f x\alpha, \Gamma \Rightarrow \Delta$.

Definition 29. A *decomposition tree of* S is a tree which expresses a process of some repeated decomposition of S.

A decomposition tree corresponds to a bottom up proof search tree of $\mathrm{F}_{16} - (\mathrm{cut})$. In every decomposition of S (i.e. S' or $S'; S''$), if S is unprovable in $\mathrm{F}_{16} - (\mathrm{cut})$, then so is S' or S''.

Lemma 30. *Let* $\Gamma \Rightarrow \Delta$ *be a given unprovable sequent in* $\mathrm{F}_{16} - (\mathrm{cut})$. *There exists an unprovable, saturated (infinite) sequent* $\Gamma^\omega \Rightarrow \Delta^\omega$ *such that* $\Gamma \subseteq \Gamma^\omega$ *and* $\Delta \subseteq \Delta^\omega$.

Proof. Let $\Gamma \Rightarrow \Delta$ be an unprovable sequent in $\mathrm{F}_{16} - (\mathrm{cut})$. We construct $\Gamma^\omega \Rightarrow \Delta^\omega$ from $\Gamma \Rightarrow \Delta$ as follows.

1. We apply the decomposition instructions from Definition 28 to $\Gamma \Rightarrow \Delta$, in the following order, but without some decomposition procedures, which are not related to the formulas in $\Gamma \Rightarrow \Delta$.

$$(1) \longrightarrow (2) \longrightarrow (3) \longrightarrow \cdots \longrightarrow (78).$$

In such a decomposition process, one of the decomposed elements S' and S'' of S is an unprovable sequent.

2. We repeat the same procedure (Step 1), infinitely often. Then, we obtain an infinite, finitely branching decomposition tree.

3. By König's lemma, we have an infinite path on this decomposition tree as follows:

$$\Gamma_0 \Rightarrow \Delta_0 \mid \Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2 \mid \cdots \infty,$$

where $\Gamma_0 \Rightarrow \Delta_0$ is $\Gamma \Rightarrow \Delta$. In this sequence of the sequents on the infinite path, we have that $\Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \cdots$ and $\Delta_0 \subseteq \Delta_1 \subseteq \Delta_2 \subseteq \cdots$.

4. We put $\Gamma^\omega := \bigcup_{i=0}^{\infty} \Gamma_i$ and $\Delta^\omega := \bigcup_{i=0}^{\infty} \Delta_i$. We note that $\Gamma^\omega \cap \Delta^\omega = \emptyset$.

Then, we have that $\Gamma \subseteq \Gamma^\omega$ and $\Delta \subseteq \Delta^\omega$, and can verify that $\Gamma^\omega \Rightarrow \Delta^\omega$ is an unprovable, saturated sequent. □

Lemma 31. *Let* $\Gamma \Rightarrow \Delta$ *be an unprovable sequent in* $\mathrm{F}_{16} - (\mathrm{cut})$, *and* $\Gamma^\omega \Rightarrow \Delta^\omega$ *be an unprovable, saturated sequent constructed from* $\Gamma \Rightarrow \Delta$ *by Lemma 30. We define a canonical model* $\mathcal{A} := \langle U, I^n, I^t, I^f, I^b \rangle$ *as follows:*

$$U := \{z \mid z \text{ is a free individual variable occurring in } \Gamma^\omega \Rightarrow \Delta^\omega\},$$
$$p^{I^n} := \{(z_1, \ldots, z_m) \mid p(z_1, \ldots, z_m) \in \Gamma^\omega\},$$
$$p^{I^t} := \{(z_1, \ldots, z_m) \mid \sim_t p(z_1, \ldots, z_m) \in \Gamma^\omega\},$$
$$p^{I^f} := \{(z_1, \ldots, z_m) \mid \sim_f p(z_1, \ldots, z_m) \in \Gamma^\omega\},$$
$$p^{I^b} := \{(z_1, \ldots, z_m) \mid \sim_b p(z_1, \ldots, z_m) \in \Gamma^\omega\}.$$

Then, for any formula α,

(1) $[(\alpha \in \Gamma^\omega \text{ implies } \mathcal{A} \models^n \underline{\alpha}) \text{ and } (\alpha \in \Delta^\omega \text{ implies not-}(\mathcal{A} \models^n \underline{\alpha}))]$,
(2) $[(\sim_t\alpha \in \Gamma^\omega \text{ implies } \mathcal{A} \models^t \underline{\alpha}) \text{ and } (\sim_t\alpha \in \Delta^\omega \text{ implies not-}(\mathcal{A} \models^t \underline{\alpha}))]$,
(3) $[(\sim_f\alpha \in \Gamma^\omega \text{ implies } \mathcal{A} \models^f \underline{\alpha}) \text{ and } (\sim_f\alpha \in \Delta^\omega \text{ implies not-}(\mathcal{A} \models^f \underline{\alpha}))]$,
(4) $[(\sim_b\alpha \in \Gamma^\omega \text{ implies } \mathcal{A} \models^b \underline{\alpha}) \text{ and } (\sim_b\alpha \in \Delta^\omega \text{ implies not-}(\mathcal{A} \models^b \underline{\alpha}))]$

where $\underline{\alpha}$ *is obtained form* α *by replacing every individual variable* x *occurring in* α *by the name* $\underline{x}$.

Proof. By (simultaneous) induction on the complexity of α.

- Base step: Obvious by the definitions of I^* ($* \in \{n, t, f, b\}$).
- Induction step for (1): We show some cases.

(Case $\alpha \equiv \beta \wedge_t \gamma$): First, we show that $\beta \wedge_t \gamma \in \Gamma^\omega$ implies $\mathcal{A} \models^n \underline{\beta} \wedge_t \underline{\gamma}$. Suppose $\beta \wedge_t \gamma \in \Gamma^\omega$. Then, we obtain $[\beta \in \Gamma^\omega$ and $\gamma \in \Gamma^\omega]$ by Definition 26. By the induction hypothesis for (1), we obtain $[\mathcal{A} \models^n \underline{\beta}$ and $\mathcal{A} \models^n \underline{\gamma}]$. This means $\mathcal{A} \models^n \underline{\beta} \wedge_t \underline{\gamma}$. Second, we show that $\beta \wedge_t \gamma \in \Delta^\omega$ implies not-($\mathcal{A} \models^n \underline{\beta} \wedge_t \underline{\gamma}$). Suppose $\beta \wedge_t \gamma \in \Delta^\omega$. Then, we obtain $[\beta \in \Delta^\omega$ or $\gamma \in \Delta^\omega]$ by Definition 26. By the induction hypothesis for (1), we obtain [not-($\mathcal{A} \models^n \underline{\beta}$) or not-($\mathcal{A} \models^n \underline{\gamma}$)]. This means not-($\mathcal{A} \models^n \underline{\beta} \wedge_t \underline{\gamma}$).

(Case $\alpha \equiv \sim_t \beta$): First, we show that $\sim_t \beta \in \Gamma^\omega$ implies $\mathcal{A} \models^n \sim_t \underline{\beta}$. Suppose $\sim_t \beta \in \Gamma^\omega$. Then we obtain $\mathcal{A} \models^t \underline{\beta}$ by the induction hypothesis for (2). Thus, we have $\mathcal{A} \models^t \underline{\beta}$, i.e., $\mathcal{A} \models^n \sim_t \underline{\beta}$. Second, we show that $\sim_t \beta \in \Delta^\omega$ implies not-($\mathcal{A} \models^n \sim_t\underline{\beta}$). Suppose $\sim_t\beta \in \Delta^\omega$. Then, we obtain not-($\mathcal{A} \models^t \underline{\beta}$) by the induction hypothesis for (2). Thus, we have not-($\mathcal{A} \models^n \sim_t\underline{\beta}$).

• Induction step for (2): We show some cases.

(Case $\alpha \equiv \beta{\rightarrow}\gamma$): First, we show that $\sim_t(\beta{\rightarrow}\gamma) \in \Gamma^\omega$ implies $\mathcal{A} \models^t \underline{\beta}{\rightarrow}\underline{\gamma}$. Suppose $\sim_t(\beta{\rightarrow}\gamma) \in \Gamma^\omega$. Then, we obtain $[\sim_t\neg\beta \in \Gamma^\omega$ and $\sim_t\gamma \in \Gamma^\omega]$ by Definition 26. By the induction hypothesis for (2), we obtain [$\mathcal{A} \models^t \neg\underline{\beta}$ and $\mathcal{A} \models^t \underline{\gamma}$], i.e., [not-($\mathcal{A} \models^t \underline{\beta}$) and $\mathcal{A} \models^t \underline{\gamma}$]. This means $\mathcal{A} \models^t \underline{\beta}{\rightarrow}\underline{\gamma}$. Second, we show that $\sim_t(\beta{\rightarrow}\gamma) \in \Delta^\omega$ implies not-($\mathcal{A} \models^- \underline{\beta}{\rightarrow}\underline{\gamma}$). Suppose $\sim_t(\beta{\rightarrow}\gamma) \in \Delta^\omega$. Then, we obtain $[\sim_t\neg\beta \in \Delta^\omega$ or $\sim_t\gamma \in \Delta^\omega]$ by Definition 26. By the induction hypothesis for (2), we obtain [not-($\mathcal{A} \models^t \neg\underline{\beta}$) or not-($\mathcal{A} \models^t \underline{\gamma}$)], i.e., [$\mathcal{A} \models^t \underline{\beta}$ or not-($\mathcal{A} \models^t \underline{\gamma}$)]. This means not-($\mathcal{A} \models^t \underline{\beta}{\rightarrow}\underline{\gamma}$).

(Case $\alpha \equiv \sim_t\beta$): First, we show that $\sim_t\sim_t\beta \in \Gamma^\omega$ implies $\mathcal{A} \models^t \sim_t\underline{\beta}$. Suppose $\sim_t\sim_t\beta \in \Gamma^\omega$. Then, we obtain $\beta \in \Gamma^\omega$ by Definition 26. By the induction hypothesis for (1) and $\beta \in \Gamma^\omega$, we obtain $\mathcal{A} \models^n \underline{\beta}$, and hence $\mathcal{A} \models^t \sim_t\underline{\beta}$. Second, we show that $\sim_t\sim_t\beta \in \Delta^\omega$ implies not-($\mathcal{A} \models^t \sim_t\underline{\beta}$). Suppose $\sim_t\sim_t\beta \in \Delta^\omega$. Then, we obtain $\beta \in \Delta^\omega$ by Definition 26. By the induction hypothesis for (1) and $\beta \in \Delta^\omega$, we obtain not-($\mathcal{A} \models^n \underline{\beta}$) and hence not-($\mathcal{A} \models^t \sim_t\underline{\beta}$).

(Case $\alpha \equiv \forall_t x\beta$): First, we show that $\sim_t\forall_t x\beta \in \Gamma^\omega$ implies $\mathcal{A} \models^t \forall_t x\underline{\beta}$. Suppose $\sim_t\forall_t x\beta \in \Gamma^\omega$. Then, we obtain $\sim_t\beta[z/x] \in \Gamma^\omega$ for some $z \in U$, by Definition 26. By the induction hypothesis for (2), we obtain that $\mathcal{A} \models^t \underline{\beta}[\underline{z}/x]$ for some $z \in U$. This means $\mathcal{A} \models^t \forall_t x\underline{\beta}$. Second, we show that $\sim_t\forall_t x\beta \in \Delta^\omega$ implies not-($\mathcal{A} \models^t \forall_t x\underline{\beta}$). Suppose $\sim_t\forall_t x\beta \in \Delta^\omega$. Then, we obtain $[\sim_t\beta[y_i/x] \in \Delta^\omega$ for all $y_i \in U]$ by Definition 26. By the induction hypothesis for (2), we obtain not-($\mathcal{A} \models^t \underline{\beta}[\underline{y_i}/x]$) for all $y_i \in U$. This means not-($\mathcal{A} \models^t \forall_t x\underline{\beta}$).

(Case $\alpha \equiv \exists_t x\beta$): First, we show that $\sim_t\exists_t x\beta \in \Gamma^\omega$ implies $\mathcal{A} \models^t \exists_t x\underline{\beta}$. Suppose $\sim_t\exists_t x\beta \in \Gamma^\omega$. Then we obtain $[\sim_t\beta[y_i/x] \in \Gamma^\omega$ for all $y_i \in U]$ by Definition 26. By the induction hypothesis, we obtain that $\mathcal{A} \models^t \underline{\beta}[\underline{y_i}/x]$ for all $y_i \in U$. This means $\mathcal{A} \models^t \exists_t x\underline{\beta}$. Second, we show that $\sim_t\exists_t x\beta \in \Delta^\omega$ implies not-($\mathcal{A} \models^t \exists_t x\underline{\beta}$). Suppose $\sim_t\exists_t x\beta \in \Delta^\omega$. Then, we obtain $[\sim_t\beta[z/x] \in \Delta^\omega$ for some $z \in U]$ by Definition 26. By the induction hypothesis for (2), we obtain not-($\mathcal{A} \models^t \underline{\beta}[\underline{z}/x]$) for some $z \in U$. This means not-($\mathcal{A} \models^t \exists_t x\underline{\beta}$).

• Induction step for (3): We show some cases.

(Case $\alpha \equiv \sim_t\beta$): First, we show that $\sim_f\sim_t\beta \in \Gamma^\omega$ implies $\mathcal{A} \models^f \sim_t\underline{\beta}$. Suppose $\sim_f\sim_t\beta \in \Gamma^\omega$, i.e., $\sim_b\beta \in \Gamma^\omega$. Then we obtain $\mathcal{A} \models^b \underline{\beta}$ by the induction hypothesis for (4). Thus, we have $\mathcal{A} \models^f \sim_t\underline{\beta}$. Second, we show that $\sim_f\sim_t\beta \in \Delta^\omega$ implies not-($\mathcal{A} \models^f \sim_t\underline{\beta}$). Suppose $\sim_f\sim_t\beta \in \Delta^\omega$, i.e., $\sim_b\beta \in \Delta^\omega$. Then, we obtain not-($\mathcal{A} \models^b \underline{\beta}$) by the induction hypothesis for (4). Thus, we have not-($\mathcal{A} \models^f \sim_t\underline{\beta}$).

(Case $\alpha \equiv \sim_b\beta$): First, we show that $\sim_f\sim_b\beta \in \Gamma^\omega$ implies $\mathcal{A} \models^f \sim_b\underline{\beta}$. Suppose $\sim_f\sim_b\beta \in \Gamma^\omega$. Then, we obtain $\sim_t\beta \in \Gamma^\omega$ by Definition 26. By the induction hypothesis for (2) and $\sim_t\beta \in \Gamma^\omega$, we obtain $\mathcal{A} \models^t \underline{\beta}$, and hence $\mathcal{A} \models^f \sim_f\sim_t\underline{\beta}$, i.e., $\mathcal{A} \models^f \sim_b\underline{\beta}$. Second, we show that $\sim_f\sim_b\beta \in \Delta^\omega$ implies not-($\mathcal{A} \models^f \sim_b\underline{\beta}$). Suppose $\sim_t\sim_b\beta \in \Delta^\omega$. Then, we obtain $\sim_t\beta \in \Delta^\omega$ by Definition 26. By the induction hypothesis for (2) and $\sim_t\beta \in \Delta^\omega$, we obtain not-($\mathcal{A} \models^t \underline{\beta}$) and hence not-($\mathcal{A} \models^f \sim_b\underline{\beta}$).

• Induction step for (4): Similar to that for (2). □

Theorem 32 (*Strong Completeness for* F_{16}). *For any sequent S, if S is valid, then* $\mathrm{F}_{16} - (\mathrm{cut}) \vdash S$.

Proof. We prove the following: if $\Gamma \Rightarrow \Delta$ is unprovable in $\mathrm{F}_{16} - (\mathrm{cut})$, then there exists a model $\mathcal{A}$ such that $\Gamma \Rightarrow \Delta$ is not valid in $\mathcal{A}$. Suppose that $\Gamma \Rightarrow \Delta$ is not provable in $\mathrm{F}_{16} - (\mathrm{cut})$. Then, by Lemma 31, we can construct a canonical model $\mathcal{A}$ satisfying the condition (1) in this lemma. Thus, we have $\mathcal{A} \models^n \underline{\gamma}$ and not-($\mathcal{A} \models^n \underline{\delta}$) for any $\gamma \in \Gamma \subseteq \Gamma^\omega$ and any $\delta \in \Delta \subseteq \Delta^\omega$. Hence, we obtain "not-($\mathcal{A} \models^n \underline{\Gamma_*}{\rightarrow}\underline{\Delta^*}$)", and hence "not-($\mathcal{A} \models^n cl(\Gamma_*{\rightarrow}\Delta^*)$)." Therefore, $\Gamma \Rightarrow \Delta$ is not valid in $\mathcal{A}$. □

Theorem 33 (*Cut-elimination for* F_{16}). *The rule* (cut) *is admissible in cut-free* F_{16}.

Proof. By combining Theorems 32 and 25. □

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